2^{nd} Partial Exam **[2h]**

1. Let $(X(t), t \geq 0)$ be a right-continuous process on a numerable set $\mathbb{S}$, starting at $i \in \mathbb{S}$. We assume that its jump chain, $(Y_n)_{n\geq 0}$ with $Y_n = X(J_n)$, is a HMC$(\delta_i, \mathbf{\Pi})$ and for each $n \geq 1$, conditional on Y_0, Y_1, ..., Y_{n-1}, the holding times $S_1, S_2, \ldots, S_n$, with $S_n = J_n - J_{n-1}$, are independent exponential random variables with parameters $q(Y_0), q(Y_1), \ldots, q(Y_{n-1})$. We define $\mathbf{Q}$ as the $\mathbf{Q}$-matrix on $\mathbb{S}$ with jump matrix $\mathbf{\Pi}$ and rate vector $(q(i), i \in \mathbb{S})$.

   Show that the matrix $\mathbf{P}(t) = (p_{ij}(t))_{i,j\in\mathbb{S}}$, with $p_{ij}(t) = \mathbb{P}_i(X(t) = j)$, is the *minimal* non-negative solution of the *Kolmogorov Backward Equations*

   $$\mathbf{P}'(t) = \mathbf{Q}\,\mathbf{P}(t), \quad \mathbf{P}(0) = \mathbb{I}\ .$$

   (3 Puntos)

2. Let $X(t) = \sigma B(t)$ be a Brownian motion with no drift and volatility $\sigma > 0$. In the formula, $B(t)$ denotes a standard Brownian motion. Compute:

   (a) the covariance function $\mathbb{C}\mathrm{ov}_X(s,t)$, for $s, t \geq 0$

   (b) the distribution of T_x, that is, the hitting time of level x, with $x > 0$.

   (3 Puntos)

3. Let $(X(t), t \geq 0)$ be a HMC on $\{0, 1, 2, \ldots\}$ with transition rates equal to

   $$\begin{aligned} q_{01} = q_{i,i+1} = 3 \quad i \geq 2 \\ q_{10} = q_{i,i-1} = 2 \quad i \geq 2 \end{aligned}$$

   and the remaining q_{ij} with $i \neq j$ equal to 0.

   Assuming that $X(0) = 4$, compute:

   (a) the probability to visit the state 0.

   (b) the limit distribution of the chain conditioned to visit the state 0 in finite time.

   (c) the $\lim_{t\to\infty} \mathbb{P}_i(X(t) = j)$ for $i, j \geq 0$.

   (4 Puntos)