Review of Matrix Algebra

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1 Basic results

Given matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ of appropriate dimensions,

1. <u>**Transposition**</u>:
$$(\boldsymbol{A}+\boldsymbol{B})' = \boldsymbol{A}' + \boldsymbol{B}' \quad (\boldsymbol{A}\boldsymbol{B})' = \boldsymbol{B}'\boldsymbol{A}'$$

2. <u>**Trace**</u>: Given a square matrix $\boldsymbol{A}$
   (a) $tr(\boldsymbol{A}) = \sum diag(\boldsymbol{A})$
   (b) $tr(k\boldsymbol{A}) = k\ tr(\boldsymbol{A}) \quad tr(\boldsymbol{A}') = tr(\boldsymbol{A})$
   (c) $tr(\boldsymbol{A}+\boldsymbol{B}) = tr(\boldsymbol{A}) + tr(\boldsymbol{B}) \quad tr(\boldsymbol{A}\boldsymbol{B}) = tr(\boldsymbol{B}\boldsymbol{A})$

3. <u>**Determinant**</u>:
   (a)
$$|\boldsymbol{A}'| = |\boldsymbol{A}| \quad |k\boldsymbol{A}| = k^n|\boldsymbol{A}| \quad |\boldsymbol{A}^{-1}| = 1/|\boldsymbol{A}|$$
   (b)
$$\begin{vmatrix} \boldsymbol{T} & \boldsymbol{U} \\ \boldsymbol{V} & \boldsymbol{W} \end{vmatrix} = |\boldsymbol{T}||\boldsymbol{W} - \boldsymbol{V}\boldsymbol{T}^{-1}\boldsymbol{U}|$$

2 Eigenvalues and eigenvectors

1. <u>**Eigenvalue**</u>: A scalar λ is said to be an eigenvalue of an $n \times n$ matrix $\boldsymbol{A}$ if there exists an $n \times 1$ nonnull vector $\boldsymbol{x}$ such that
$$\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}$$
then $\boldsymbol{x}$ is an eigenvector of $\boldsymbol{A}$

2. Given an $n \times n$ matrix $\boldsymbol{A}$ with distintic eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_k$, with multiplicities $\gamma_1, \gamma_2, \cdots \gamma_k$
   (a) $Rank(\boldsymbol{A})$ equals the number of nonzero eigenvalues
   (b) $tr(\boldsymbol{A}) = \sum_{i=1}^{k} \gamma_i \lambda_i$
   (c) $det(\boldsymbol{A}) = \prod_{i=1}^{k} \lambda_i^{\gamma_i}$

3 Inverse Matrices

1. <u>**Inverse of a matrix**</u>: Let $\boldsymbol{A}$ be a $k \times k$ matrix. The inverse of $\boldsymbol{A}$, $\boldsymbol{A}^{-1}$ is another $k \times k$ matrix such that
$$\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$$

2. <u>**Generalized inverse of a matrix**</u>: A generalized inverse of an $m \times n$ matrix $\boldsymbol{A}$ is any $n \times m$ matrix $\boldsymbol{G}$ such that
$$\boldsymbol{A}\boldsymbol{G}\boldsymbol{A} = \boldsymbol{A}$$

3. **Inverse of a sum of matrices**:

$$(\boldsymbol{R}+\boldsymbol{STU})^{-1} = \boldsymbol{R}^{-1} - \boldsymbol{R}^{-1}\boldsymbol{S}(\boldsymbol{T}^{-1}+\boldsymbol{U}\boldsymbol{R}^{-1}\boldsymbol{S})^{-1}\boldsymbol{U}\boldsymbol{R}^{-1}$$

4. **Inverse of a partitioned matrix**:

$$\begin{pmatrix} \boldsymbol{T} & \boldsymbol{U} \\ \boldsymbol{V} & \boldsymbol{W} \end{pmatrix} = \begin{pmatrix} \boldsymbol{T}^{-1}+\boldsymbol{T}^{-1}\boldsymbol{U}\boldsymbol{Q}^{-1}\boldsymbol{V}\boldsymbol{T}^{-1} & -\boldsymbol{T}^{-1}\boldsymbol{U}\boldsymbol{Q}^{-1} \\ -\boldsymbol{Q}^{-1}\boldsymbol{V}\boldsymbol{T}^{-1} & \boldsymbol{Q}^{-1} \end{pmatrix}$$

where $\boldsymbol{Q} = \boldsymbol{W} - \boldsymbol{V}\boldsymbol{T}^{-1}\boldsymbol{U}$

4 Special Matrices

1. **Non-negative definite and positive define matrices**:

A real symmetric matrix $\boldsymbol{A}$

(a)

$$\begin{aligned} \text{non-negative definite} &\Leftrightarrow x'\boldsymbol{A}x \geq 0 \text{ for all } x \\ &\Leftrightarrow \text{all eigenvalues of } \boldsymbol{A} \text{ are } \geq 0 \end{aligned}$$

(b)

$$\begin{aligned} \text{positive definite} &\Leftrightarrow x'\boldsymbol{A}x > 0 \text{ for all } x \neq 0 \\ &\Leftrightarrow \text{all eigenvalues of } \boldsymbol{A} \text{ are } > 0 \\ &\Leftrightarrow \text{is non-singular} \end{aligned}$$

2. **Singular matrix**: A $n \times n$ matrix is singular if $Rank(\boldsymbol{A}) < n$

3. **Idempotent matrix**: Let $\boldsymbol{A}$ be a $k \times k$ matrix, $\boldsymbol{A}$ is idempotent if $\boldsymbol{AA} = \boldsymbol{A}$

4. **Orthogonal matrix**: A square matrix $\boldsymbol{A}$ is orthogonal if

$$\boldsymbol{A}'\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}' = \boldsymbol{I}$$

if $\boldsymbol{A}$ is non-singular $\boldsymbol{A}' = \boldsymbol{A}^{-1}$

5 Matrix decomposition

1. **Diagonalization of a symmetric matrix**: Let $\boldsymbol{A}$ be an $n \times n$ symmetric matrix, then

$$\boldsymbol{PAP}' = diag(\lambda_i)$$

where λ_i are the eigenvalues of $\boldsymbol{A}$ and $\boldsymbol{P}$ is the orthogonal matrix with columns equal to the eigenvectors of $\boldsymbol{A}$

2. **$\boldsymbol{QR}$ decomposition**: A $m \times n$ matrix $\boldsymbol{A}$ with $Rank(\boldsymbol{A}) = n$ may be decomposed as

$$\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{R}$$

where $\boldsymbol{Q}$ is orthogonal and $\boldsymbol{R}$ is an upper triangular matrix with positive diagonal elements

3. **Cholesky decomposition**: A symmetric, positive definite matrix $\boldsymbol{A}$ may be decomposed as

$$\boldsymbol{A} = \boldsymbol{L}\boldsymbol{L}'$$

where $\boldsymbol{L}$ is an lower triangular matrix with positive diagonal elements

4. **Singular value decomposition**: The singular value decomposition of a $m \times n$ matrix is given by:

$$\boldsymbol{A} = \boldsymbol{V} \begin{pmatrix} diag_r(\lambda_i) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix} \boldsymbol{U}'$$

where $r = Rank(\boldsymbol{A})$, $\boldsymbol{U}$ and $\boldsymbol{V}$ are orthogonal matrices and λ_i^2 are the non-zero eigenvalues of $\boldsymbol{A}\boldsymbol{A}'$

6 Matrix derivatives

Let $\boldsymbol{A}$ be a $k \times k$ matrix of constants, $\boldsymbol{a}$ a $k \times 1$ vector of constants and $\boldsymbol{y}$ a vector of variables:

1. $$\frac{\partial \boldsymbol{a}'\boldsymbol{y}}{\partial \boldsymbol{y}} = \boldsymbol{a}$$

2. $$\frac{\partial \boldsymbol{y}'\boldsymbol{y}}{\partial \boldsymbol{y}} = 2\boldsymbol{y}$$

3. $$\frac{\partial \boldsymbol{a}'\boldsymbol{A}\boldsymbol{y}}{\partial \boldsymbol{y}} = \boldsymbol{A}'\boldsymbol{a}$$

4. $$\frac{\partial \boldsymbol{y}'\boldsymbol{A}\boldsymbol{y}}{\partial \boldsymbol{y}} = \boldsymbol{A}\boldsymbol{y} + \boldsymbol{A}'\boldsymbol{y}$$

7 Expectations and Variances

Let $\boldsymbol{A}$ be a $k \times k$ matrix of constants, $\boldsymbol{a}$ a $k \times 1$ vector of constants and $\boldsymbol{y}$ a random vector with mean $\boldsymbol{\mu}$ and variance-covariance matrix $\boldsymbol{V}$

1. $E(\boldsymbol{a}'\boldsymbol{y}) = \boldsymbol{a}'\boldsymbol{\mu}$
2. $E(\boldsymbol{A}\boldsymbol{y}) = \boldsymbol{A}\boldsymbol{\mu}$

3. $Var(\boldsymbol{a}'\boldsymbol{y}) = \boldsymbol{a}'\boldsymbol{V}\boldsymbol{a}$

4. $Var(\boldsymbol{A}\boldsymbol{y}) = \boldsymbol{A}\boldsymbol{V}\boldsymbol{A}'$. Note that if $\boldsymbol{V} = \sigma^2\boldsymbol{I}$, then $Var(\boldsymbol{A}\boldsymbol{y}) = \sigma^2\boldsymbol{A}\boldsymbol{A}'$

5. $E(\boldsymbol{y}'\boldsymbol{A}\boldsymbol{y}) = tr(\boldsymbol{A}\boldsymbol{V}) + \boldsymbol{\mu}'\boldsymbol{A}\boldsymbol{\mu}$

8 Distributions

1. Let $Y_i, Y_2, \ldots, Y_n$ be independent norrmally distributed random variables with $E(Y_i) = \mu_i$ and $Var(Y_i) = \sigma_i^2$. Let $a_1, a_2, \ldots, a_n$ be known constants. Then,

$$\boldsymbol{U} = \sum_{i=1}^{n} a_i Y_i \sim N\left(\sum_{i=1}^{n} a_i\mu_i, \sum_{i=1}^{n} a_i^2\sigma_i^2\right)$$

2. If $Y \sim N(\mu, \sigma^2)$, then,

$$Z = \frac{Y-\mu}{\sigma} \sim N(0,1) \quad Z^2 \sim \chi_1^2$$

3. Let $Y_i, Y_2, \ldots, Y_n$ be independent norrmally distributed random variables with $E(Y_i) = \mu_i$ and $Var(Y_i) = \sigma_i^2$, and let$Z_i = \frac{Y_i-\mu_i}{\sigma_i}$ then,

$$\sum_{i=1}^{n} Z_i^2 \sim \chi_n^2$$

4. if $Z \sim N(0,1)$ and, $V \sim \chi_g^2$, and Z and V are independent, then

$$\frac{Z}{\sqrt{V/g}} \sim t_g$$

5. Let $V \sim \chi_{g_1}^2$, and $W \sim \chi_{g_2}^2$. If V and W are independent, then

$$\frac{V/g_1}{W/g_2} \sim F_{g_1,g_2}$$