Comments

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1 Sum of independent random variables

Given two independent random variables X and Y, with distribution functions $F_X(x) = \Pr\{X \leq x\}$ and $F_Y(y) = \Pr\{Y \leq y\}$ if we define $Z = X + Y$, the distribution function $F_Z(z)$ is the given by

$$\begin{aligned} F_Z(z) = \Pr\{Z \leq z\} = \Pr\{X + Y \leq z\} &= \int_{-\infty}^{\infty} \Pr\{X + Y \leq z | X = x\}\, dF_X(x) \\ &= \int_{-\infty}^{\infty} \Pr\{Y \leq z - x | X = x\}\, dF_X(x) \\ &\overset{\text{ind}}{=} \int_{-\infty}^{\infty} \Pr\{Y \leq z - x\}\, dF_X(x) \\ &= \int_{-\infty}^{\infty} F_Y(z - x)\, dF_X(x) \end{aligned} \tag{1}$$

And if we define the convolution operator $F*G(t)$ that given two distribution functions $F(t)$, $G(t)$ is equal to

$$F * G(t) = \int_{-\infty}^{\infty} F(t - s)\, dG(s)$$

we get that

$$F_Z(z) = F_Y * F_X(z).$$

You can check that $F * G(t) = G * F(t)$ by a simple change of variable or simply noting that $Z = X + Y = Y + X$.

2 Positive random variables

Assuming now that $X, Y \geq 0$, we have that $F_X(x) = 0$ for $x < 0$ and also $F_Y(y) = 0$ for $y < 0$. Note that we admit $F_X(0) > 0$ and $F_Y(0) > 0$. In this case equation (1) becomes

$$F_Z(z) = \int_0^z F_Y(z - x)\, dF_X(x) = \int_0^z F_X(z - y)\, dF_Y(y) \tag{2}$$

still it is valid that $F_Z(z) = F_Y * F_X(z) = F_X * F_Y(z)$ that now reduces to the expression (2).

2.1 Laplace transforms

Assume that X has Laplace transform $\phi_X(s) = \tilde{F}_X(s) = \mathbb{E}[e^{s\,X}]$, and in the same way Y has Laplace transform $\phi_Y(s) = \tilde{F}_Y(s) = \mathbb{E}[e^{s\,Y}]$, then it follows that $Z = X + Y$ has Laplace transform $\phi_Z(s) = \tilde{F}_Z(s)$ given by

$$\begin{aligned} \tilde{F}_Z(s) &= \mathbb{E}[e^{s\,Z}] = \mathbb{E}[e^{s\,(X+Y)}] = \mathbb{E}[e^{s\,X}\, e^{s\,Y}] \\ &\overset{\text{ind}}{=} \mathbb{E}[e^{s\,X}]\mathbb{E}[e^{s\,Y}] = \tilde{F}_X(s)\, \tilde{F}_Y(s) \end{aligned}$$

Last equation expresses the fact that the Laplace transform translates the convolution operator to a product operator, i.e.

$$\mathcal{L}(F * G(t)) = \mathcal{L}(F(t))\,\mathcal{L}(G(t)),$$

where we denoted by $\mathcal{L}(F(t))$ the Laplace-Stieltjes transform of the distribution function $F(t)$, i.e.

$$\mathcal{L}(F(t)) = \int_0^\infty e^{-s\,t}\,dF(t).$$

3 Sum of i.i.d. positive random variables

Assume to have two i.i.d. random variables X_1 and X_2 with common distribution function $F_X(x)$. Now $Y = X_1 + X_2$ has distribution function

$$F_Y(y) = F_X * F_X(y) = F_X^{*2}(y),$$

known also has 2-fold convolution of $F_X(x)$. In general if $Z = \sum_{i=1}^n X_i$ with X_i i.i.d. and with common distribution $F_X(x)$, then the distribution of Z is given by the n-th convolution of $F_X(x)$, i.e.

$$F_Z(z) = F_X^{*n}(z)$$

with $F_X^{*n}(z) = F_X^{*(n-1)} * F_X(z)$, that in the transformed domain corresponds to

$$\phi_Z(s) = \tilde{F}_Z(s) = \left(\tilde{F}_X(s)\right)^n.$$

4 Excercises

1. Assume X_1 and X_2 i.i.d uniformly distributed between $[0, 1]$, compute the distribution function of $Y = X_1 + X_2$. Compute its Laplace transform as well.

2. Assume X_n i.i.d exponential random variables with parameter $\lambda > 0$. Compute the distribution of $Z = \sum_{i=1}^n X_i$. Do you recognize this distribution? Compute its Laplace transform as well.

3. If $\Phi_(z)$ is the CDF of a standard Normal random variable. What would be its n-th fold convolution?