Chapter III: Probability - Exercises

Bernardo D'Auria

Statistics Department

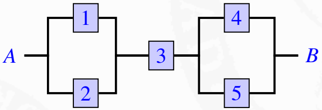
Universidad Carlos III de Madrid

GROUP 89 - COMPUTER ENGINEERING

2010-2011

Exercice

We have a system of connected components according to the following figure:



All components are of similar reliability and have a probability of failure of 0.01. The failures of one component are independent of the state of other components. The system is functioning if you can find between A and B a path of components operating. **What is the probability that the system works?**

Exercice

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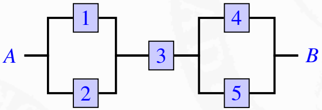

All components are of similar reliability and have a probability of failure of 0.01. The failures of one component are independent of the state of other components. The system is functioning if you can find between A and B a path of components operating.
What is the probability that the system works?

SOLUTION:

$$\Pr(\text{The system works}) = 0.9898$$

Exercise

A person wants to get money from an ATM machine but s/he does not remember last two digits of the secret code. S/he decides to guess it by randomly typing them. The ATM analyze each digit separately, that means that if one digit is incorrectly typed it asks the user to type all the secret code again from the beginning. A user has only 3 possibility to type correctly her/his secret code.

a) What is the probability that the person succeed in typing randomly her/his code?

b) If s/he make it, what is the probability that s/he was able to type the correct code in her/his third trial?

Solution - a)

What is the probability that the person succeed in typing randomly her/his code?

Denote by R_B the event *typing correctly the one BEFORE last digit* and by W_L *typing incorrectly the LAST digit*, we have that

$$\begin{aligned}
\Pr\{\text{correctly guess}\} = & \Pr\{R_B\, R_L\} + \Pr\{R_B\, W_L\, R_L\} + \Pr\{R_B\, W_L\, W_L\, R_L\} \\
& \Pr\{W_B\, R_B\, R_L\} + \Pr\{W_B\, R_B\, W_L\, R_L\} \\
& \Pr\{W_B\, W_B\, R_B\, R_L\}
\end{aligned}$$

Solution - a)

What is the probability that the person succeed in typing randomly her/his code?

Denote by R_B the event *typing correctly the one BEFORE last digit* and by W_L *typing incorrectly the LAST digit*, we have that

$$\Pr\{\text{correctly guess}\} = \boxed{\Pr\{R_B\, R_L\}} + \Pr\{R_B\, W_L\, R_L\} + \Pr\{R_B\, W_L\, W_L\, R_L\}$$
$$\Pr\{W_B\, R_B\, R_L\} + \Pr\{W_B\, R_B\, W_L\, R_L\}$$
$$\Pr\{W_B\, W_B\, R_B\, R_L\}$$

$$\Pr\{R_B\, R_L\} = \Pr\{R_B\}\Pr\{R_L\} = \frac{1}{10}\,\frac{1}{10} = \frac{1}{100}$$

Solution - a)

What is the probability that the person succeed in typing randomly her/his code?
Denote by R_B the event *typing correctly the one BEFORE last digit* and by W_L *typing incorrectly the LAST digit*, we have that

$$\begin{aligned}\Pr\{\text{correctly guess}\} = \Pr\{R_B\, R_L\} + \boxed{\Pr\{R_B\, W_L\, R_L\}} + \Pr\{R_B\, W_L\, W_L\, R_L\}\\ \Pr\{W_B\, R_B\, R_L\} + \Pr\{W_B\, R_B\, W_L\, R_L\}\\ \Pr\{W_B\, W_B\, R_B\, R_L\}\end{aligned}$$

$$\begin{aligned}\Pr\{R_B\, W_L\, R_L\} &= \Pr\{R_B\}\Pr\{W_L\, R_L\}\\ &= \frac{1}{10}\Pr\{W_L\}\Pr\{R_L|W_L\} = \frac{1}{10}\frac{9}{10}\frac{1}{9} = \frac{1}{100}\end{aligned}$$

Solution - a)

What is the probability that the person succeed in typing randomly her/his code?

Denote by R_B the event *typing correctly the one BEFORE last digit* and by W_L *typing incorrectly the LAST digit*, we have that

$$\begin{aligned}\Pr\{\text{correctly guess}\} = {} & \Pr\{R_B\,R_L\} + \Pr\{R_B\,W_L\,R_L\} + \boxed{\Pr\{R_B\,W_L\,W_L\,R_L\}} \\ & \Pr\{W_B\,R_B\,R_L\} + \Pr\{W_B\,R_B\,W_L\,R_L\} \\ & \Pr\{W_B\,W_B\,R_B\,R_L\}\end{aligned}$$

$$\begin{aligned}\Pr\{R_B\,W_L\,W_L\,R_L\} &= \Pr\{R_B\}\Pr\{W_L\,W_L\,R_L\} = \frac{1}{10}\Pr\{W_L\}\Pr\{W_L\,R_L|W_L\} \\ &= \frac{1}{10}\frac{9}{10}\Pr\{W_L|W_L\}\Pr\{R_L|W_LW_L\} = \frac{1}{10}\frac{9}{10}\frac{8}{9}\frac{1}{8} = \frac{1}{100}\end{aligned}$$

Solution - a)

What is the probability that the person succeed in typing randomly her/his code?

Denote by R_B the event *typing correctly the one BEFORE last digit* and by W_L *typing incorrectly the LAST digit*, we have that

$$
\begin{aligned}
\Pr\{\text{correctly guess}\} &= \Pr\{R_B\, R_L\} + \Pr\{R_B\, W_L\, R_L\} + \Pr\{R_B\, W_L\, W_L\, R_L\} \\
&\Pr\{W_B\, R_B\, R_L\} + \Pr\{W_B\, R_B\, W_L\, R_L\} \\
&\Pr\{W_B\, W_B\, R_B\, R_L\} \\
&= \frac{1}{10}\frac{1}{10} + \frac{1}{10}\frac{9}{10}\frac{1}{9} + \frac{1}{10}\frac{9}{10}\frac{8}{9}\frac{1}{8} \\
&\quad + \frac{9}{10}\frac{1}{9}\frac{1}{10} + \frac{9}{10}\frac{1}{9}\frac{9}{10}\frac{1}{9} \\
&\quad + \frac{9}{10}\frac{8}{9}\frac{1}{8}\frac{1}{10} = \boxed{\frac{6}{100}}
\end{aligned}
$$

Solution - a)

What is the probability that the person succeed in typing randomly her/his code?

Using a different method, we have that

$$\begin{aligned}\Pr\{\overline{\text{correctly guess}}\} &= \Pr\{W_B\, W_B\, W_B\} + \Pr\{R_B\, W_L\, W_L\, W_L\} \\ &\quad + \Pr\{W_B\, R_B\, W_L\, W_L\} + \Pr\{W_B\, W_B\, R_B\, W_L\} \\ &= \frac{9}{10}\frac{8}{9}\frac{7}{8} + \frac{1}{10}\frac{9}{10}\frac{8}{9}\frac{7}{8} \\ &\quad + \frac{9}{10}\frac{1}{9}\frac{9}{10}\frac{8}{9} + \frac{9}{10}\frac{8}{9}\frac{1}{8}\frac{9}{10} = \frac{94}{100}\end{aligned}$$

and eventually

$$\Pr\{\text{correctly guess}\} = 1 - \Pr\{\overline{\text{correctly guess}}\} = \boxed{\frac{6}{100}}$$

Solution - b)

What is the probability that s/he was able to type the correct code in her/his third trial?

$$\begin{aligned}\Pr\{\text{guess at the } 3^{\text{rd}} \text{ trial}\} &= \Pr\{R_B\,W_L\,W_L\,R_B\} + \Pr\{W_B\,R_B\,W_L\,R_B\} \\ &\quad + \Pr\{W_B\,W_B\,R_B\,R_L\} \\ &= \frac{1}{10}\frac{9}{10}\frac{8}{9}\frac{1}{8} + \frac{9}{10}\frac{1}{9}\frac{9}{10}\frac{1}{9} + \frac{9}{10}\frac{8}{9}\frac{1}{8}\frac{1}{10} = \frac{3}{100}\end{aligned}$$

Therefore

$$\begin{aligned}\Pr\{\text{guess at the } 3^{\text{rd}} \text{ trial}\} &= \frac{\Pr\{\text{guess at the } 3^{\text{rd}} \text{ trial, guess correctly}\}}{\Pr\{\text{guess correctly}\}} \\ &= \frac{3}{100}\frac{100}{6} = \boxed{\frac{1}{2}}\end{aligned}$$

Exercise

An electric component is shipped in boxes made of 10 units. The set is discarded if after inspecting it it is found a not working component. The inspection only test at most two components of the full set. The inspection can be done in two ways

- The inspector use a machine to make the test. S/he simultaneously introduces the 2 electric components to be tested and reject the set if at least one is defective.
- The inspector follows the following procedure: S/he tests one component, if it is defective s/he rejects the set. If the first component is good s/he tests a second one. If the second one is good s/he accepts the set as good otherwise s/he rejects the set.

If we know that the set has 4 defective components, **what is the probability of rejecting the set for the two different inspecting processes?**

Exercise

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If we know that the set has 4 defective components, **what is the probability of rejecting the set for the two different inspecting processes?**

SOLUTION:

$$\Pr(R_1) = \Pr(R_2) = \frac{2}{3}$$

Exercise - Ing. Teleco February 2006 - C1

Using a transmission channel, we can transmit one of the following three messages:

$$M_1 = 1111 \quad M_2 = 2222 \quad M_3 = 3333$$

with probabilities $p_1 = 0.5$, $p_2 = 0.25$ and $p_3 = 0.25$.
Each transmitted digit is correctly received with probability α and with probability $(1 - \alpha)$ it is incorrectly received, i.e with probability 50% it is received one of the other remaining two digits .

a) What is the probability to receive the message $M = 1231$?
b) What is the probability that having received the message M, the sent message was M_1?

Solution

$a)$ *What is the probability to receive the message* $M = 1231$*?*

$$\Pr\{M\} = \Pr\{M|M_1\}\Pr\{M_1\} + \Pr\{M|M_2\}\Pr\{M_2\} + \Pr\{M|M_3\}\Pr\{M_3\}$$

$$= \alpha\left(\frac{1-\alpha}{2}\right)\left(\frac{1-\alpha}{2}\right)\alpha p_1 + \left(\frac{1-\alpha}{2}\right)^3 \alpha p_2 + \left(\frac{1-\alpha}{2}\right)^3 \alpha p_3$$

$$= \alpha^2\left(\frac{1-\alpha}{2}\right)^2 p_1 + \left(\frac{1-\alpha}{2}\right)^3 \alpha(p_2+p_3)$$

$$= \alpha\left(\frac{1-\alpha}{2}\right)^2 (\alpha p_1 + \frac{1-\alpha}{2} p_1) = \alpha\left(\frac{1-\alpha}{2}\right)^2 \frac{2\alpha p_1 + p_1 - \alpha p_1}{2}$$

$$= \alpha\left(\frac{1-\alpha}{2}\right)^2 \frac{\alpha+1}{4}$$

$b)$ *What is the probability that having received the message* M*, the sent message was* M_1*?*

$$\Pr\{M_1|M\} = \frac{\Pr\{M_1|M\}}{\Pr\{M\}} = \frac{\Pr\{M|M_1\}\Pr\{M_1\}}{\frac{\alpha}{2}\left(\frac{1-\alpha}{2}\right)^2} = \frac{\alpha^2\left(\frac{1-\alpha}{2}\right)^2\frac{1}{2}}{\alpha\left(\frac{1-\alpha}{2}\right)^2\frac{\alpha+1}{4}} = \frac{2\alpha}{\alpha+1}$$

Exercise - Ing. Teleco September 2005 - C2

We have two urns. The urn U_1 has 70% of white balls and 30% of black balls, the urn U_2, has 30% of white balls and 70% of black balls. We randomly choose one urn and keep 10 balls from, each time replacing the chosen ball again in the urn.
The result is: $B = wbwwwwbwww$, where w denotes “white” ball and b denotes “black” ball.

What is the probability that we have chosen the urn U_1?

Solution $1/2$

We select a urn, since there are 2 of them and we choose it randomly we have:

$$\Pr\{U_1\} = \Pr\{U_2\} = 0.5$$

The event B is made by the intersection of 10 independent events, since we replace the ball after having taken it from the urn. We have

$$\Pr\{w|U_1\} = 0.7 \quad \text{and} \quad \Pr\{b|U_1\} = 0.3$$

and that

$$\begin{aligned}\Pr\{B|U_1\} &= \Pr\{wbwwwwbwww|U_1\} \\ &= \Pr\{w|U_1\} \cdot \Pr\{b|U_1\} \cdot \Pr\{w|U_1\} \cdots \Pr\{w|U_1\} \\ &= (\Pr\{w|U_1\})^8 \, (\Pr\{b|U_1\})^2 = 0.7^8 \times 0.3^2\end{aligned}$$

Solution $2/2$

In the same way

$$\Pr\{B|U_2\} = 0.3^8 \times 0.7^2$$

The required probability is $\Pr\{U_1|B\}$.

Using Bayes' Theorem we get:

$$\begin{aligned}
\Pr\{U_1|B\} &= \frac{\Pr\{B|U_1\}\Pr\{U_1\}}{\Pr\{B|U_1\}\Pr\{U_1\} + \Pr\{B|U_2\}\Pr\{U_2\}} \\
&= \frac{0.7^8 \times 0.3^2 \times 0.5}{0.7^8 \times 0.3^2 \times 0.5 + 0.3^8 \times 0.7^2 \times 0.5} \\
&= 99.4\%
\end{aligned}$$