Tema 2: Programación Lineal

Bernardo D'Auria

Departamento de Estadística

Universidad Carlos III de Madrid

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Ejercicio

Resolver el siguiente problema de programación linear

$$
\begin{array}{rrl}
\text{max} & x_1 + 2x_2 & \\
\text{sujeto a} & -2x_1 + x_2 & \leq 2 \\
 & -x_1 + 2x_2 & \leq 7 \\
 & x_1 & \leq 3 \\
 & \boldsymbol{x} & \geq 0
\end{array}
$$

Las alternativas viven en un espacio de dimensión 2, es decir $\boldsymbol{x} \in \mathbb{R}^2$.

SOLUCIÓN Paso 0

Forma estándar

$$
\begin{array}{rrl}
\text{min} & -x_1 - 2x_2 & \\
\\
\text{sujeto a} & -2x_1 + x_2 + x_3 & = 2 \\
 & -x_1 + 2x_2 + x_4 & = 7 \\
 & x_1 + x_5 & = 3 \\
 & \boldsymbol{x} & \geq 0
\end{array}
$$

Las alternativas viven en un espacio de dimensión 5, es decir $\boldsymbol{x} \in \mathbb{R}^5$.

$$
\boldsymbol{A} = \begin{pmatrix} -2 & 1 & 1 & 0 & 0 \\ -1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \boldsymbol{b} = \begin{pmatrix} 2 \\ 7 \\ 3 \end{pmatrix} \quad \boldsymbol{c} = \begin{pmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

SIMPLEX — Paso 1.1

$$\boldsymbol{I} = \{3,4,5\}, \quad \boldsymbol{J} = \{1,2\}$$

$$\boldsymbol{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \; \boldsymbol{c_B} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \; \boldsymbol{N} = \begin{pmatrix} -2 & 1 \\ -1 & 2 \\ 1 & 0 \end{pmatrix} \; \boldsymbol{c_N} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

$$\boldsymbol{x_B} = \boldsymbol{B}^{-1}\boldsymbol{b} = \begin{pmatrix} 2 \\ 7 \\ 3 \end{pmatrix} ; \boldsymbol{x_N} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} ; \boldsymbol{x} = \begin{pmatrix} 0 & 0 & 2 & 7 & 3 \end{pmatrix}^t$$

$$\boldsymbol{\lambda} = (\boldsymbol{B}^t)^{-1}\boldsymbol{c_B} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \boldsymbol{\sigma} = \boldsymbol{c_N} - \boldsymbol{N}^t\boldsymbol{\lambda} = \begin{pmatrix} -1 \\ -2 \end{pmatrix} \quad \boxed{\boldsymbol{\sigma} \not\geq 0}$$

SIMPLEX — Paso 1.2

$$\boxed{j = 2} \quad \boldsymbol{p_B} = -\boldsymbol{B}^{-1} \boldsymbol{N}_2 = \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix} \quad \boldsymbol{p_N} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \boxed{\boldsymbol{p_B} \not\geq 0}$$

$$\alpha = \min_{i \in \{1,2\}} \left\{ 2, \frac{7}{2} \right\} = 2; \quad \boldsymbol{p}^t = \begin{pmatrix} 0 & 1 & -1 & -2 & 0 \end{pmatrix} \quad \boxed{i = 1}$$

$$\boldsymbol{x}^+ = \boldsymbol{x} + \alpha \boldsymbol{p} = \begin{pmatrix} 0 & 2 & 0 & 3 & 3 \end{pmatrix}^t; \quad \boldsymbol{c}^t \cdot \boldsymbol{x}^+ = -4$$

$$\boxed{j = 2,\, i = 1} \quad \boldsymbol{I}_{[\boldsymbol{I}_i \leftrightarrow \boldsymbol{J}_j]} \rightarrow \boldsymbol{I}^+ = \{2, 4, 5\}; \quad \boldsymbol{J}_{[\boldsymbol{J}_j \leftrightarrow \boldsymbol{I}_i]} \rightarrow \boldsymbol{J}^+ = \{1, 3\};$$

SIMPLEX Paso 2.1

$$I = \{2,4,5\}, \quad J = \{1,3\}; \quad x = \begin{pmatrix} 0 & 2 & 0 & 3 & 3 \end{pmatrix}^t$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \; c_B = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} \; N = \begin{pmatrix} -2 & 1 \\ -1 & 0 \\ 1 & 0 \end{pmatrix} \; c_N = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\lambda = (B^t)^{-1} c_B = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} \; \sigma = c_N - N^t \lambda = \begin{pmatrix} -5 \\ 2 \end{pmatrix} \quad \boxed{\sigma \not\geq 0}$$

SIMPLEX Paso 2.2

$$\boxed{j=1} \quad \boldsymbol{p_B} = -\boldsymbol{B}^{-1}\boldsymbol{N}_1 = \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} \quad \boldsymbol{p_N} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \boxed{\boldsymbol{p_B} \not\geq 0}$$

$$\alpha = \min_{i\in\{2,3\}} \left\{ \frac{3}{3}, \frac{3}{1} \right\} = 1; \quad \boldsymbol{p}^t = \begin{pmatrix} 1 & 2 & 0 & -3 & -1 \end{pmatrix} \quad \boxed{i=2}$$

$$\boldsymbol{x}^+ = \boldsymbol{x} + \alpha\boldsymbol{p} = \begin{pmatrix} 1 & 4 & 0 & 0 & 2 \end{pmatrix}^t; \quad \boldsymbol{c}^t \cdot \boldsymbol{x}^+ = -9$$

$$\boxed{j=1,\, i=2} \quad \boldsymbol{I}_{[\boldsymbol{I}_i \leftrightarrow \boldsymbol{J}_j]} \rightarrow \boldsymbol{I}^+ = \{2, 1, 5\}; \quad \boldsymbol{J}_{[\boldsymbol{J}_j \leftrightarrow \boldsymbol{I}_i]} \rightarrow \boldsymbol{J}^+ = \{4, 3\};$$

SIMPLEX Paso 3.1

$$\boldsymbol{I} = \{2, 1, 5\}, \quad \boldsymbol{J} = \{4, 3\}; \quad \boldsymbol{x} = \begin{pmatrix} 1 & 4 & 0 & 0 & 2 \end{pmatrix}^t$$

$$\boldsymbol{B} = \begin{pmatrix} 1 & -2 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \boldsymbol{c_B} = \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix} \quad \boldsymbol{N} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \boldsymbol{c_N} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\boldsymbol{B}^{-1} = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & 0 \\ -\frac{2}{3} & \frac{1}{3} & 0 \\ \frac{2}{3} & -\frac{1}{3} & 1 \end{pmatrix}$$

$$\boldsymbol{\lambda} = (\boldsymbol{B}^t)^{-1}\boldsymbol{c_B} = \begin{pmatrix} \frac{4}{3} \\ -\frac{5}{3} \\ 0 \end{pmatrix} \quad \boldsymbol{\sigma} = \boldsymbol{c_N} - \boldsymbol{N}^t\,\boldsymbol{\lambda} = \begin{pmatrix} \frac{5}{3} \\ -\frac{4}{3} \end{pmatrix} \quad \boxed{\boldsymbol{\sigma} \ngeq 0}$$

SIMPLEX — Paso 3.2

$$\boxed{j=2} \quad \boldsymbol{p_B} = -\boldsymbol{B}^{-1}\boldsymbol{N}_2 = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{pmatrix} \quad \boldsymbol{p_N} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \boxed{\boldsymbol{p_B} \not\geq 0}$$

$$\alpha = \min_{i \in \{3\}} \{3\} = 3; \quad \boldsymbol{p}^t = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 1 & 0 & -\frac{2}{3} \end{pmatrix} \quad \boxed{i=3}$$

$$\boldsymbol{x}^+ = \boldsymbol{x} + \alpha\,\boldsymbol{p} = \begin{pmatrix} 3 & 5 & 3 & 0 & 0 \end{pmatrix}^t; \quad \boldsymbol{c}^t \cdot \boldsymbol{x}^+ = -13$$

$$\boxed{j=2,\, i=3} \quad \boldsymbol{I}_{[\boldsymbol{I}_i \leftrightarrow \boldsymbol{J}_j]} \rightarrow \boldsymbol{I}^+ = \{2, 1, 3\}; \quad \boldsymbol{J}_{[\boldsymbol{J}_j \leftrightarrow \boldsymbol{I}_i]} \rightarrow \boldsymbol{J}^+ = \{4, 5\};$$

SIMPLEX Paso 4.1

$$\boldsymbol{I} = \{2, 1, 3\}, \quad \boldsymbol{J} = \{4, 5\}; \quad \boldsymbol{x} = \begin{pmatrix} 3 & 5 & 3 & 0 & 0 \end{pmatrix}^t$$

$$\boldsymbol{B} = \begin{pmatrix} 1 & -2 & 1 \\ 2 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \boldsymbol{c_B} = \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix} \boldsymbol{N} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \boldsymbol{c_N} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\boldsymbol{B}^{-1} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \\ 1 & -\frac{1}{2} & \frac{3}{2} \end{pmatrix}$$

$$\boldsymbol{\lambda} = (\boldsymbol{B}^t)^{-1}\boldsymbol{c_B} = \begin{pmatrix} 0 \\ -1 \\ -2 \end{pmatrix} \boldsymbol{\sigma} = \boldsymbol{c_N} - \boldsymbol{N}^t\boldsymbol{\lambda} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \boxed{\boldsymbol{\sigma} \geq 0}$$

FIN $x_1 = 3 \quad x_2 = 5 \quad \max = -\boldsymbol{c}^t \cdot \boldsymbol{x} = 13$

SIMPLEX   Solución Geometrica