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Bounded-rational-prisoners' dilemma: On critical phenomena of cooperation

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Abstract

Models based on agents has been widely used to study the paradigmatic emergence of cooperation in social systems. In the spatial prisoners' dilemma game introduced by Martin Nowak and Robert May, the agents are greedy and imitate indiscriminately the action of the wealthiest neighbor. Other strategies, for example stochastic or pavlovian, have also been considered showing similar results as the greedy rule. That is, in general matter, the asymptotic emergence or maintenance of cooperation. For these spatial models, it can be proved, and that is the dilemma, that cooperation extinguishes when agents exhibit completely rational behavior. In this work, we explore the behavior of a system with bounded-rational agents. For that, we consider a modified spatial prisoners' dilemma on an adaptive network where each agent can play different actions with different neighbors. The coevolutive dynamic obeys a scheme of rational imitation of the wealthiest agent of each neighborhood. We show the existence of a phase transition (absence of cooperation–presence of cooperation) that depends on the incentive to defect. We compute the critical value and report a simulation study that evidences that the emergence or survival of cooperation at the steady state is a critical phenomenon. These results provide a fascinating point of view to understand the trade-off between cooperation and rationality in wealthy societies. Our results also include the emergence of a rich social structure living in asymptotic regimen. Throughout a simulation study, we analyze the distribution of wealth and other complex aspects of the social network at the steady state.

Keywords: Prisoners' dilemma; Adaptive networks; Rational-bounded agents

1. Introduction

The spatial prisoners' dilemma leads to the fundamental problem of collective action: the emergence of cooperation in social systems. Addressing this problem, Nowak and May [1] introduced a simple model based on *greedy* agents on a regular lattice. At each time step, the agents recollect pay-offs from the interactions with their neighbors, according to a prisoners' dilemma pay-off matrix, and play, at the next time step, the same action played by the neighbor with the highest score. This model displays a rich spatiotemporal dynamic

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and it has been widely used in computational simulations to study the complexity of cooperation (see also [2–5]).

Eguíluz et al. [6] go beyond these results by introducing an adaptive rule in which each agent is able to change his neighborhood (*social plasticity*). Throughout a simulation study, they show new results, including asymptotic steady states with high proportion of cooperators, and the emergence of a hierarchical network that governs the global dynamic of the system (social structure).

The greedy agent of Nowak and May model acts irrationally. He takes the decision to cooperate or defect based only on the last action of the successful agent of his neighborhood, with no assessment on how this action affects his own benefit. He takes this action to all his neighbors, no matter what they do. On the other hand, if we assume, as correspond from a game theory approach, that agents are completely rational, in an economic sense, then it is easy to see that the system reaches the absorbing state of non-cooperation in one time step.

The aim of this work is to analyze the asymptotic behavior of the system when there exists a trade-off between the greedy action of imitating the successful agent and the rational action of maximizing the profit. For this, we must consider bounded-rational agents. Our approach is based on a modified spatial prisoners' dilemma in which each agent can take different actions with different neighbors. This multidimensional strategy setting involves a new definition of what a cooperative agent is. Here, an agent is cooperative if the number of agents with whom he cooperates is greater or equal than the number of agents that cooperate with him. By incorporating plasticity, we model an artificial social system by an adaptive network and we prove that there exists a phase transition (absence of cooperation–presence of cooperation) that depends on the incentive to defect. We compute the critical value for the incentive and prove by simulations that the emergence or survival of cooperation at the steady state is a critical phenomenon. Our results also include the emergence of a rich social structure in asymptotic regimen, where the leaders are cooperative agents that interact with their neighbors according to a tit-for-tat strategy. The presence of free-riders and the exclusion of defectors are other novelties observed. Next, we present a full description of our model, which we call bounded-rational-prisoners' dilemma.

2. The bounded-rational-prisoners' dilemma

We follow the standard approach where a social system is modeled by arranging individuals on a network. Each node represents an individual or agent and two nodes are connected by a link when there exists an interaction between them. In that case, we say agents are neighbors. The neighborhood of an agent is composed by those agents directly connected to him. We model the evolutionary dynamic of the system by implementing synchronously two main rules at each time step:

- A strategy rule, in which each agent decides how to interact with his neighbors.
- An adaptive rule, in which the neighborhood of each agent can be modified.

Let $\mathcal{N} = \{1, 2, \ldots, N\}$ be the set of all agents. Denote by $\mathcal{N}_i$ the neighborhood of agent i and let Γ_i be its cardinality. For each $j \in \mathcal{N}_i$, the variable S_{ij} will represent the action of i with j. If agent i cooperates with j then $S_{ij} = 1$, otherwise $S_{ij} = 0$. In subsequent,

$$S_{i\cdot} \equiv \sum_{j \in \mathcal{N}_i} S_{ij} \quad \text{and} \quad S_{\cdot i} \equiv \sum_{j \in \mathcal{N}_i} S_{ji}. \tag{1}$$

Thus, $S_{i\cdot}$ represents the number of neighbors with whom agent i cooperates and $S_{\cdot i}$ represents the number of neighbors that cooperate with agent i. All previous studies of the spatial prisoners' dilemma have focussed on two singular cases $S_{i\cdot}$ is either Γ_i or 0. There, the characteristic or state of being either a cooperator or a defector is exclusively defined by the own indiscriminate action taking by the agent. Contrarily, we will assume that agents are able to decide with whom they are going to cooperate or to defect. In this more general and realistic case, $S_{i\cdot}$ can take any value of $\{0, 1, \ldots, \Gamma_i\}$. Here, it is required a new definition of what a cooperative agent is. Among the wide range of possibilities, we consider an innovative concept with social perspective, since the characteristic or state of being either a cooperator or a defector relies on a relation between $S_{i\cdot}$ and $S_{\cdot i}$. In other words, the state of an agent does not depend solely on how he behaves with his neighborhood but also how the neighborhood behave with him.

Definition 1. We say that agent i is *cooperator* if he satisfies the two following conditions:

1. $S_{i\cdot} \neq 0$, that is, the cooperation of i with his neighborhood is no null.
2. $S_{i\cdot} \geqslant S_{\cdot i}$, that is, i cooperates more with his neighbors than his neighbors cooperate with him.

If an agent is non-cooperative, we say that he is a *defector*. Note that, according to our definition, any indiscriminate cooperator is always a cooperator in our context and also any indiscriminate defector is always a defector, no matter how his neighbors behave with him. Thus, our definition is, in particular, a generalization of the standard concept of cooperator for the model with multidimensional strategies that we are considering.

Let $P_{ij} \equiv P(S_{ij}, S_{ji})$ the pay-off of agent i from his interaction with j. According to a prisoners' dilemma pay-off matrix, $P(0,1) \equiv T$ is the temptation or incentive to defect, $P(1,1) \equiv R$ is the reward for mutual cooperation, $P(0,0) \equiv P$ is the punishment for mutual defection, and, we will assume without lost of generality, $P(1,0) \equiv 0$. Thus, we set $T > R > P > 0$ and $2R > T$. The profit or total pay-off recollected by agent i is

$$\Pi_i = \sum_{j \in \mathcal{N}_i} P_{ij} \tag{2}$$

and *successful agents* are those that receive the highest profit from the interactions with their neighbors. The non-successful agents are called *unsatisfied agents*. In general, we say that an agent is greedy if he changes his state to the state of the successful neighbor. It is important to note that this greedy rule can be performed by more than one set of actions. In particular, the irrational greedy agent of Nowak–May model changes his state by becoming either indiscriminate cooperator or indiscriminate defector. Now we introduce what we consider a bounded-rational agent.

Definition 2. *A rational greedy agent* is a greedy agent that changes his state by choosing the optimal set of actions that maximizes his profit.

In other words, a cooperative agent i is rational greedy if he imitates a successful defector by solving the following optimization problem:

$$\max \Pi_i \quad \text{subject to} \quad S_{i\cdot} < S_{\cdot i}, \quad \text{or} \quad S_{i\cdot} = 0. \tag{3}$$

When agent i is a defector, he is rational greedy if he imitates a successful cooperator by solving the optimization problem:

$$\max \Pi_i \quad \text{subject to} \quad S_{i\cdot} \geqslant S_{\cdot i} \quad \text{and} \quad S_{i\cdot} \neq 0. \tag{4}$$

We compute at the Appendix explicit and simple solutions for (3) and (4) in terms of variables S_{ji}, $j \in \mathcal{N}_i$. Similar to the pure greedy strategy of the Nowak–May model, the rational greedy rule involves no memory and no forecasting. The agent action is deterministic and is based only on interactions with his neighbors, according to the behavior of a wide range of individuals we observe in real life. As optimizers, rational agents imitate a defector adopting an indiscriminate defective state. However, when they imitate a cooperator they adopt a *tit-for-tat state*, when $T < R + P$, and defect with all possible cooperative neighbors, when $T > R + P$. The bounded-rational-prisoners' dilemma relies on agents described above. We are providing a simple setting to extend the spatial prisoners' dilemma when greedy agent are able to decide with whom they are going to cooperate or not.

Following [6], the plasticity of the social system is modeled by an network dynamic in which defective and unsatisfied agents can break a link with a defective neighbor and rewire it with someone else. The main difference here settles on what a defector is. The adaptive rule considered for local changes within neighborhoods is based on our concept of defection. We will assume that, with probability p, each defective and unsatisfied agent will break his interactions with each defective neighbor, no matter their particular pairwise interaction. Broken links are replaced by new ones with agents randomly chosen. The new actions involved in this proce-

Table 1
Evolutionary dynamic of rational greedy agents

		Successful	
		Cooperator	Defector
Unsatisfied	Cooperator	The unsatisfied agent does not change	The cooperative agent changes his state
	Defector	The defective agent changes his state and updates his neighborhood	The unsatisfied agent updates his neighborhood

dure come defined by agent states. Thus, if i breaks his link with j and replaces it with a link with k, then $S_{ik} = S_{ij}$ and S_{ki} will be either 0 or 1 depending on the state of agent k. This adaptive rule is simple and has a strong social component, since agents break links taking in account the whole behavior of agents with their neighbors and neglecting their own particular interactions with those agents. Forward we will discuss additional aspects about the plasticity defined here that complement our analysis.

Table 1 resumes the evolutionary dynamic of the model.

3. Cooperation enhancement as a critical phenomenon

Previous studies based on irrational greedy agents consider a simplified version of the spatial dilemma given by the pay-off matrix with $R = 1$ and $P \to 0$. Lindgren and Nordhal [3] found that there is not noticeable difference by considering $0 \leqslant P < 0.1$. The temptation T, the incentive to defect, is thus the only parameter in these models. There, the fraction of cooperators F_C in asymptotic regime is studied only as a function of T. In broad terms, for non-adaptive networks F_C fluctuates around an average value f_C that decreases as the incentive T increases and defectors dominate the network for highest values of temptation. This transition has been studied in detail for the two-dimensional lattice by Nowak and May [3] and by Scheweitzer et al. [7] for more general fixed networks. In contrast, Zimmermann et al. [5] show by an extensive simulation study that $f_C > 0.8$ for adaptive networks. Plasticity on irrational greedy agents promotes higher level of cooperation in spite of the incentive to defect. Higher values of T have a slight effect over the fraction of cooperators which remains significantly high. In addition, on adaptive networks the fraction of cooperators at time t, $F_C(t)$, reaches a steady state when $t \to +\infty$.

It is notorious that increasing values of T promote free-riders and $P \to 0$ is a severe punishment that weakens mutual defection. Considering this facts, we point out that cooperation enhancement should be studied as function of T and P for a fixed parameter R. Both T and P may be considered incentives to defect. We try to come closer to more realistic social systems in which individuals can be susceptible to changes in these incentives. Besides, over all possible pairwise interactions (C–C, C–D, D–C, D–D), plasticity on irrational greedy agents only removes D–D links rising the number of neighbors of cooperative agents. In consequence plasticity promotes straightforward the emergence of successful cooperators when random initial networks have a sufficiently high proportion of cooperators. Plasticity on rational greedy agents can remove any kind of link, since the adaptive rule takes in account collective behavior of agents. Our multi-dimensional approach leads the analysis of a more complete framework to study the asymptotic behavior of cooperation for different values of parameters T and P and plasticity as a social norm.

We consider random initial networks with L links uniformly distributed among N agents. Let $K = E(\Gamma_i)$ be the average of links per agent. Each link is chosen among all pairwise interactions with the same probability. We checked as in [5] that $p < 1$ only sets different time scales for the evolution of the network and state updates. For asymptotic analysis we can assume without loss of generality $p = 1$, which represents the simultaneous update of states and neighborhoods. The dynamical behavior of the system depends on the relation among the parameters T, P and R. This relation brings the condition to go from a cooperative phase to a defective phase. As we mentioned in previous section, the solutions of the maximization problems (3) and (4) allow characterize the states of rational greedy agents depending on whether T is greater or smaller than $R + P$. The knowledge about these states brings light to conjecture

$$T > R + P \quad \text{implies} \quad f_C \approx 0 \quad \text{and} \quad T < R + P \quad \text{implies} \quad f_C \gg 0 \tag{5}$$

for large N and large initial connectivity. Moreover, we are able to prove that the system achieves the absorbing state of full indiscriminate defection for $T > R + P$, except for some stationary initial configurations. We observe (5) by means of computer simulations. Having in mind models on the square regular lattice, we choose $N = 100 \times 100$ and $K = 4$ for our simulations. We average over 100 random initial networks with $L = 20000$ to estimate f_C. The results for $R = 2$ are shown in Fig. 1.

We characterize numerically the discontinuity at the critical point $T = R + P$ by proving that $f_C > 0.64$ on $T < R + P$. The lower bound 0.64 was estimated by extensive simulation for $T \to R$ as $P \to 0$. Fig. 2 illustrates the reported critical phenomenon by showing the average of the fractions of cooperators f_C as a function of T with $R = 2$ and $P = 1.4$.

Other finding is the paradoxical behavior of the wealth of the system at asymptotic regime when the parameter T varies. We illustrate this by averaging the mean profit per agent over all initial networks and graphic it as a function of T. We can observe how this average value π rises as T increases spite of f_C decreases, see Fig. 2. This happens because defectors are well connected with cooperators and they take advantage of this to free-ride. The incentive to keep free-riding works successfully until T reaches the critical value $T = R + P$ where a social catastrophe occurs. For $T > R + P$, the average value π achieves the lowest constant value P. This last

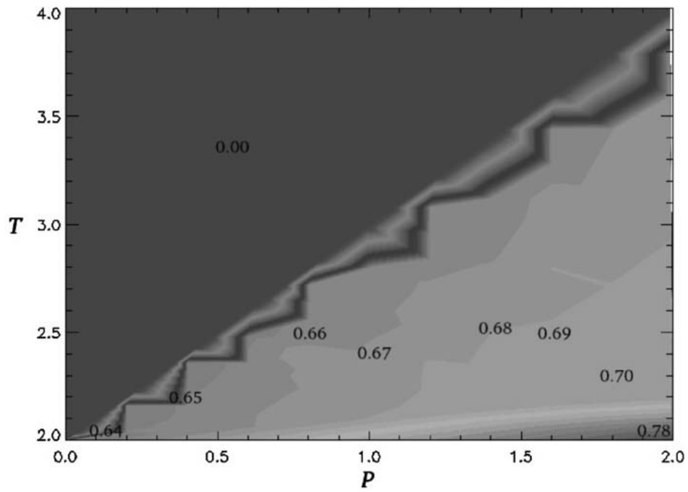

Fig. 1. Average levels of cooperators in asymptotic regime f_C ($R = 2$).

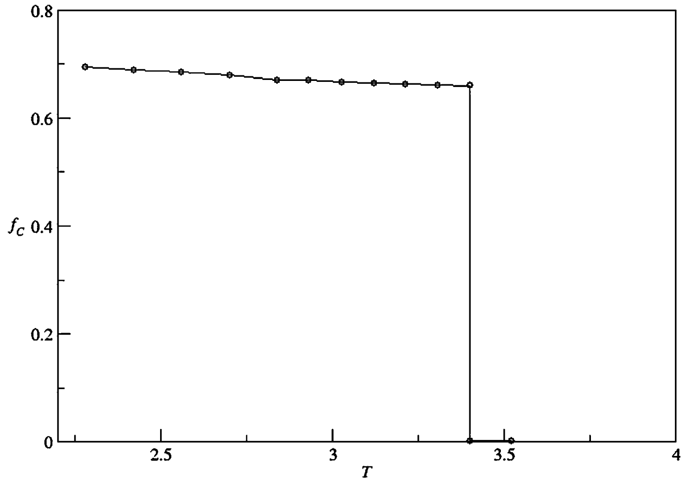

Fig. 2. Expected fraction of cooperators in the steady state f_C ($R = 2$, $P = 1.4$).

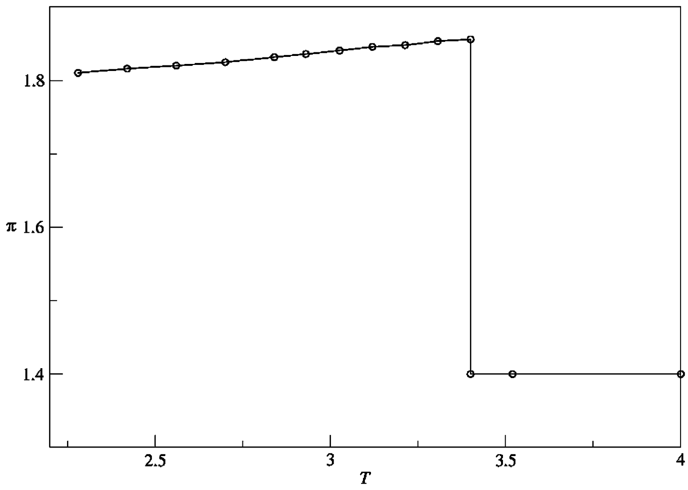

Fig. 3. Expected mean profit per agent π in the stationary regime ($R = 2$, $P = 1.4$).

result is a straightforward consequence of the fact that the system achieves in asymptotic regime the absorbing state of full indiscriminate defection (see Fig. 3).

4. Distribution of wealth and social structure

In order to understand how the network structure is in stationary cooperative phase, we introduce the benevolence of agent i by $\Delta_i = S_{i\cdot} - S_{\cdot i}$. Thus, we say that i is a benevolent cooperator if $\Delta_i > 0$. If $\Delta_j < 0$ the defector j is called free-rider. Null benevolence can be reached by both cooperators and defectors, however in cooperative phase null benevolence is an exclusive characteristic of cooperators in tit-for-tat state or indiscriminate defectors without D–C links.

We examined networks in steady state from more than 6500 random initial configurations, different values of T and P in cooperative phase ($T < R + P$) and $R = 2$. This extensive simulation permits us characterize some aspects of the social structure in asymptotic regime. First at all, we must remark that agents with higher profits are always tit-for-tat cooperators. That is, cooperators with only C–C and D–D links. The successful of these agents is based on their high connectivities. Figs. 4 and 5 show these aspects for a typical realization.

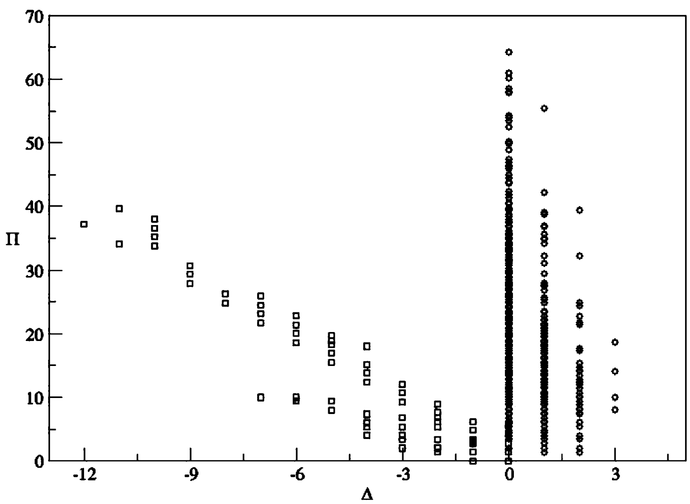

Fig. 4. Benevolence Δ and profit Π per agent in the steady state for a random initial network ($T = 3.1$, $R = 2$, $P = 1.4$, (○): cooperators, (□): defectors).

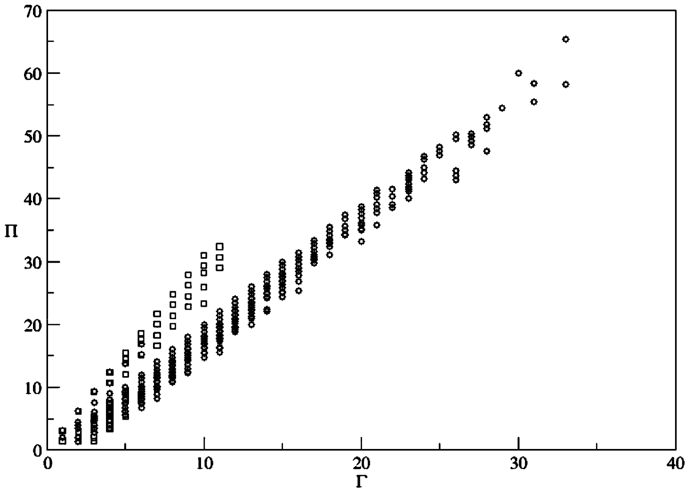

Fig. 5. Number of neighbors Γ and profit Π per agent in the stationary state for a random initial network ($T = 3.1$, $R = 2$, $P = 1.4$, (○): cooperators, (□): defectors).

Other remarkable characteristic we observed in all cases was the emergence of both free-riders and benevolent agents. Free-riders donot profit more than the mean profit of cooperators because the social dynamic punishes them with low connectivity.

The system displays a rich social structure in which we can observe isolated indiscriminate defectors, free-riders, conformist cooperator, tit-for-tat leaders and successful benevolent agents.

5. Concluding remarks

Despite the important pioneer results obtained in previous studies, it is impossible to assure that in real social system individuals behave identical one with each others inside their neighborhoods and even more it is unrealistic to assume that individuals are either complete rational or complete irrational. To complete, within a social perspective the characteristic for being a cooperator or defector is not a subject matter that depends solely on the actions of the individual involved but also depends on the actions chosen by his neighbors. For other side, the plasticity in irrational greedy agents works as a straightforward incentive to cooperate since it reinforces the free-ride interactions and weakens the mutual defection. The aim of this work is to address this issues in order to come closer to a more realistic simulation of a social system. First, we consider agents capable to select with whom in his neighborhood they are going to cooperate. This obliges to redefine the concept of what a cooperative individual is. We introduce an innovate definition with a strong social content. Second, we consider greedy individuals that are rational in the selection of his new actions. Finally, the plasticity implemented in our model is performed within a social context since individuals break relations with neighbors according to their characteristic or state instead of their own particular links or interactions. All these considerations let us analyze the conditions required to observe the emergence and maintenance of cooperation in such systems. We observe among other findings; phase transitions, critical phenomena and social macrostructures. To conclude, we point out how the system is susceptible to change when the incentives to defect increases becoming more and more wealthy. This happens until the incentive reaches the critical value, after that a social catastrophe occurs.

Appendix

Rational greedy imitation from a cooperator to a defector: The optimization problem (3) has trivial optimal solution $S^*_{ij} = 0$, for all $j \in \mathcal{N}_i$. This is an easy consequence of $T > R$ and $P > 0$. That means, the cooperative agent is rational greedy if he imitates a defective agent becoming *indiscriminately defective*; that is, he becomes defector with all in his neighborhood.

Rational greedy imitation from a defector to a cooperator: Since agent i imitates a cooperative neighbor, we can suppose that at least one neighbor is a cooperator. Then, the condition $S_{i\cdot} \neq 0$ is satisfied and problem (4) is reduced to

$$\max \Pi_i \quad \text{subject to} \quad S_{i\cdot} \geqslant S_{\cdot i}. \tag{6}$$

The solution to this problem is easily represented if we distinguish between the two following cases:

1. $T < R + P$, which has the optimal solution the strategy tat-for-tit $S^*_{ij} = S_{ji}$.
2. $T > R + P$, for which it is convenient to distinguish the following subcases:
   (a) If $S_{\cdot i} = \Gamma_i - S_{\cdot i}$ then the solution to the problem is to take "the opposite action"

   $$S^*_{ij} = 0 \quad \text{if} \quad S_{ji} = 1 \quad \text{and} \quad S^*_{ij} = 1 \quad \text{if} \quad S_{ji} = 0.$$

   (b) If $S_{\cdot i} < \Gamma_i - S_{\cdot i}$ then the solution to the problem is

   $$S^*_{ij} = 0 \quad \text{if} \quad S_{ji} = 1 \quad \text{and} \quad S^*_{ij} = \varepsilon_k \quad \text{if} \quad S_{ji} = 0,$$

   where $\{\varepsilon_1, \ldots, \varepsilon\Gamma_i - S_{\cdot i}\}$ is a permutation of $\Gamma_i - 2S_{\cdot i}$ Zero's and $S_{\cdot i}$ One's.
   (c) If $S_{\cdot i} > \Gamma_i - S_{\cdot i}$ then the solution to the problem is

   $$S^*_{ij} = \varepsilon_k \quad \text{if} \quad S_{ji} = 1 \quad \text{and} \quad S^*_{ij} = 1 \quad \text{if} \quad S_{ji} = 0,$$

   where $\{\varepsilon_1, \ldots, \varepsilon_{S\cdot i}\}$ is a permutation of $\Gamma_i - S_{\cdot i}$ Zero's and $2S_{\cdot i} - \Gamma_i$ One's.

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