Notes on how to obtain informal asymptotics from Renewal-Reward-Regenerative Processes

Following the last two slides of the lectures, if $\{N(t)\}$ is a Renewal Process with arrival times $S_1, S_2, \ldots$ and $\{X(t)\}$ is a regenerative process with cycles $C_n = S_n - S_{n-1}$.

Then, considering the "rewards"

$$R_n = \int_{S_{n-1}}^{S_n} X(t)\, dt$$

usually we can derivate

$$(1) \qquad \frac{1}{t}\int_0^t X(s)\, ds \longrightarrow \frac{E(R_i)}{E(C_i)}$$

In addition, from (1), usually we can derivate

$$(2) \qquad \frac{1}{t}\int_0^t E(X(s))\, ds \longrightarrow \frac{E(R_i)}{E(C_i)}$$

Moreover, under bounding conditions,

Namely $\bullet$ if $\lim_{t\to\infty} E(X(t))$ exists.

$\bullet$ $\forall_t \int_0^t E(X(s))ds < \infty$

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t E(X(s))ds = \lim_{t\to\infty} E(X(t)).$$

Putting all together, we obtain a powerful tool to prove limit theorems related with this type of processes.

Example 1 Alternating Renewals.

System is alternating between ON and OFF

ON-durations are iid $U_1, U_2, \ldots$ } the sequences

OFF-durations are iid $D_1, D_2, \ldots$ } are independent

n^{th}-cycle is $C_n = U_n + D_n$

$$X(t) = \begin{cases} 1 & \text{if the system is ON at time } t \\ 0 & \text{otherwise} \end{cases}$$

$R_n = U_n$ so $\frac{1}{t}\int_0^t x(s)\,ds \to \frac{\mathbb{E}(U_i)}{\mathbb{E}(U_i)+\mathbb{E}(D_i)}$

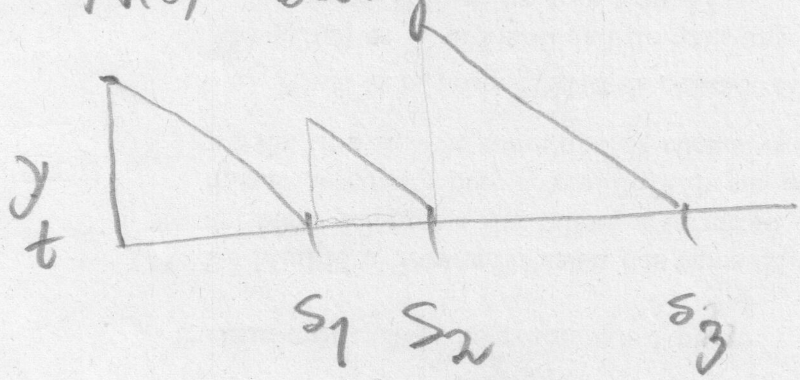
Note $P(X(s)=1) = \mathbb{E}(X(s))$

$$\longrightarrow \frac{\mathbb{E}(U_i)}{\mathbb{E}(U_i)+\mathbb{E}(D_i)}$$

Example 2 Asymptotics for the excess life (slide 20)

$$\gamma_t = S(N(t)+1) - t$$

N(t) being a Renewal Process.



Derivation of $\mathbb{E}(\gamma_t) \longrightarrow \frac{\mathbb{E}(S_1^2)}{2\,\mathbb{E}(S_1)}$

$X(t) = \gamma_t$ $\quad R_n = \int_{S_{n-1}}^{S_n} \gamma_t \, dt \overset{d}{=} \int_0^{S_1} \gamma_t \, dt$

$$= \int_0^{S_1} (S_1 - t)\, dt = \frac{S_1^2}{2}$$

$$\frac{1}{t}\int_0^t \gamma_s \, ds \longrightarrow \frac{E(S_1^2)}{2\,E(S_1)}.$$

then, it follows $E(\gamma_t) \to \dfrac{E(S_1^2)}{2\,E(S_1)}.$

Derivation of $P(\gamma_t \leq x) \to \dfrac{1}{\mu}\displaystyle\int_0^x P(S_1 > y)\,dy$

$$\boxed{\mu = E(S_1)} \qquad X(t) = \begin{cases} 1 & \text{if } \gamma_t \leq x \\ 0 & \text{otherwise} \end{cases}$$

$$R_n = \int_{S_{n-1}}^{S_n} \mathbb{1}_{(0,x]}(\gamma_t)\,dt \overset{\mathcal{L}}{=} \int_0^{S_1} \mathbb{1}_{(0,x]}(\gamma_t)\,dt$$

$$= \min(S_1, x)$$

$$\mathbb{E}(\min(S_1,x)) = \int_0^{+\infty} P(\min(S_1,x) > y)\,dy$$

$$= \int_0^x P(\min(S_1,x) \geq y)\,dy = \int_0^x P(S_1 > y)\,dy$$

$$\Longrightarrow \frac{1}{t}\int_0^t X(s)\,ds \to \frac{1}{\mu}\int_0^x P(S_1 > y)\,dy$$