Stochastic Process: Exercise # 5

TA Session

April 4th, 2013

Problem 1 *Bin Packing*

For the following martingale questions we set the preliminary notations:

$$\mathbb{E}[Y \mid X_1, X_2, \ldots, X_n] \equiv \mathbb{E}[Y \mid \mathscr{F}_n]$$

Let $\{L_n, n = 1, 2, \ldots\}$ be a random variable sequence, denote

$$\mathbb{E}[L_n \mid \mathscr{F}_i] \equiv \mathbb{E}[L_n \mid X_1, X_2, \ldots, X_i] \equiv Y_i$$

Then $\{Y_i, i = 1, 2, \ldots\}$ is a martingale with respect to $\{X_1, X_2, \ldots\}$. In particular,

$$\mathbb{E}[Y_i \mid \mathscr{F}_{i-1}] = Y_{i-1}$$

The bin packing problem is a basic problem of operations research. Given n objects with sizes $x_1, x_2, \ldots, x_n$, and an unlimited collection of bins each of size 1. In the randomized version of this problem, we suppose that the objects have independent random sizes $X_1, X_2, \ldots$ having some common distribution on $[0, 1]$. Let B_n be the random number of bins required in order to pack $X_1, X_2, \ldots, X_n$ efficiently.

(a) What is the minimum number of bins required in order to pack the objects?

(b) How close B_n is to its mean value $\mathbb{E}(B_n)$?

Problem 2 *Prove the following exercises. Maximal Inequality*

1. If $(Y, \mathscr{F})$ is a submartingale and $Y_n^* \equiv \max\{Y_i, 0 \le i \le n\}$, then

$$\mathbb{P}(Y_n^* \ge x) \le \frac{\mathbb{E}(Y_n^+)}{x} \quad \text{for} \quad x > 0 \tag{1}$$

2. If $(Y, \mathscr{F})$ is a supermartingale and $\mathbb{E} \mid Y_0 \mid < \infty$, then

$$\mathbb{P}(Y_n^* \ge x) \le \frac{\mathbb{E}(Y_0) + \mathbb{E}(Y_n^-)}{x} \quad \text{for} \quad x > 0 \tag{2}$$

Problem 3 *An Autoregressive Process.*

Let $Z_0, Z_1, \ldots,$ be uncorrelated random variables with $\mathbb{E}[Z_n] = 0, n \geq 0$ and

$$Var(Z_n) = \begin{cases} \sigma^2/(1-\lambda^2), & n = 0 \\ \sigma^2, & n \geq 1 \end{cases}$$

where$\lambda^2 < 1$. Define

$$X_0 = Z_0$$
$$X_n = \lambda X_{n-1} + Z_n, n \geq 1$$

The process $\{X_n, n \geq 0\}$ is called *first-order autoregressive process.* It says that the state at time n(that is X_n) is a constant multiple of the state at time $n-1$ plus a random error term Z_n. Find $Cov(X_n, X_{n+m})$.

Problem 4 *Solve the following exercises.*

Let $X(t)$ be a standard Brownian motion and define $Y(t) = tX(1/t)$.

(a) What is the distribution of $Y(t)$?

(b) Compute $\text{Cov}(Y(s), Y(t))$.

(c) Argue that $\{Y(t), t \geq 0\}$ is also Brownian motion.

(d) Let $T = \inf\{t > 0, X(t) = 0\}$. Using (c) present an argument that $\mathbb{P}\{T = 0\} = 1$.

Problem 5 *Prove the following exercises.*

Let $X(t)$ be a standard Brownian motion and define $W(t) = X(a^2t)/a$ for $a > 0$. Verify that $W(t)$ is also Brownian motion.

Problem 6 *Prove the following exercise.*

A stochastic process $\{X(t), t \geq 0\}$ is said to be stationary if $X(t_1), \ldots, X(t_n)$ has the same joint distribution as $X(t_1 + a), \ldots, X(t_n + a)$ for all $n, a, t_1, \ldots, t_n$.

(a) Prove that a Gaussian process is stationary iff $\text{Cov}(X(s), X(t))$ depends only on $t - s, s \leq t$, and $\mathbb{E}[X(t)] = c$.

(b) Let $\{X(t), t \geq 0\}$ be Brownian motion and define

$$V(t) = e^{-\alpha t/2} X(\alpha e^{\alpha t})$$

Show that $\{V(t), t \geq 0\}$ is a stationary Gaussian process. It is called the Ornstein-Uhlenbeck process.