Stochastic Process: Exercise # 4

TA Session

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Problem 1 *On-Off System*

Whenever a machine of a given type is "operative", it stays that way for a random length of time with exponential density with mean $1/\lambda$. When a breakdown occurs, repairs lasting for a random length of time with exponential density with mean $1/\mu$ are started immediately. Repairs return the machine again to the operative state. Let $X(t)$ be the state of the machine at time t.

Suppose the state space is $S = \{0, 1\}$ with 0 corresponding to the operative state and 1 corresponding to the system being under repair. We have that $X(t)$ is a Markov chain with state space S. Find the transition matrix $\mathbb{P}(t)$.

Problem 2 *Harry's Basketball Injuries*

Harry's illustrious semi-probasketball career war marred by the fact that he was injury prone. During his playing career he fluctuated between three states: 0 (fit), 1 (minor injury not preventing competition), 2 (major injury requiring complete abstinence from the court). The team doctor, upon observing the transitions between states, concluded these transitions could be modeled by a Markov chain with transition matrix

$$Q = \begin{pmatrix} 0 & 1/3 & 2/3 \\ 1/3 & 0 & 2/3 \\ 1 & 0 & 0 \end{pmatrix}$$

Holding times in states $0, 1, 2$ were exponentially distributed with parameters $\frac{1}{3}, \frac{1}{3}, \frac{1}{6}$ respectively.

Let $X(t)$ be the continuous time Markov chain describing the state of health of our hero at time t. (Assume the time units are days.)

(1) What is the generator matrix?

(2) What is the long run proportion of time that our hero had to abstain from playing?

(3) Harry was paid \$40/day when fit, \$30/day when able to play injured and \$20/day when he was injured and unable to play. What was the long run earning rate of our hero?

Problem 3 *A Linear Growth model with Immigration*

A model in which

$$\mu_n = n\mu,\ n \geq 1$$
$$\lambda_n = n\lambda + \theta,\ n \geq 0$$

is called a linear growth process with immigration. Such processes occur naturally in the study of biological reproduction and population growth. Each individual in the population is assumed to give birth at an exponential rate λ; in addition, there is an exponential rate of increase θ of the population due to an external source such as immigration. Hence, the total birth rate where there are n persons in the system is $n\lambda + \theta$. Deaths are assumed to occur at an exponential rate μ for each member of the population, so $\mu_n = n\mu$.

Let $X(t)$ denote the population size at time t. Suppose that $X(0) = i$ and let $M(t) = \mathbb{E}[X(t)]$, Find $M(t)$.

Problem 4 *M/M/1 Queue with a Finite Waiting Room.*

Suppose arrivals occur according to a poison process rate a and service lengths are iid, independent of the arrivals with distribution $1 - e^{-bx}, x > 0$. The wrinkle here is that an arriving customer who finds two customers in the system departs immediately without waiting for service, and the interpretation is that the capacity of the system is two or the waiting room only seats one.

Let $Q(t)$ be the number in the system at time t, so that $S = \{0, 1, 2\}$. Find $\mathbb{P}_{20}(t)$.

Problem 5 *A Machine Repair Model*

Consider a job shop that consists of M machines and one serviceman. Suppose that the amount of time each machine runs before breaking down is exponentially distributed with mean $1/\lambda$, and suppose that the amount of time that it takes for the serviceman to fix a machine is exponentially distributed with mean $1/\mu$. We shall attempt to answer these questions:

(a) What is the average number of machines not in use?

(b) What proportion of time is each machine in use?