Stochastic Process: Exercise # 2

TA Session

February 26th, 2013

Problem 1 *Solve the following exercises.*

Consider a Markov Chain X_n with its finite state space $\{0, 1, 2\}$ and let X_n evolves as a Markov chain with transition matrix given as follows:

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Assume the initial state is $X_0 = 0$, please find $\mathbb{P}(X_n = 0|X_0 = 0)$.

Problem 2 *Theorem 3.1 in Page 40 in the Durret's book*

We say that T is a stopping time if the occurence(or nonoccurrence) of the event "we stop at time n",$\{T = n\}$ can be determined by looking at the values of the process up to that time: $X_0, \ldots, X_n$. Show the following theorem:

Suppose T is a stopping time. Given that $T = n$ and $X_T = y$, any other information about $X_0, \ldots, X_T$ is irrelevant for predicting the future, and $X_{T+k}, k \geq 0$ behaves like the Markov chain with initial state y.

Problem 3 *Amateur Night at Happy Harry's*

Friday night is amateur night at Happy Harry's Restaurant where a seemingly infinite stream of performers dreaming of stardom perform in lieu of the usual professional floor show. The quality of the performers fall into five categories with "1" being the best and '5' being unspeakably atrocious, representing for Harry's discriminating clientele an excedance of the threshold of pain which may cause a riot. The probability a class 5 performer will cause the crowd to riot is .3. After the riot is quelled, performances resume-the show must go on. Since performers tend to bring along friends of similar talent to perform, it is found that the succession of states on Friday night at Happy Harry's can be modelled as a six-state Markov chain where state 6 represents 'riot' and state 'i' represents a class 'i' performer, $1 \leq i \leq 5$. The transition matrix for this chain is

$$\begin{pmatrix} .05 & .15 & .3 & .3 & .2 & 0 \\ .05 & .3 & .3 & .3 & .05 & 0 \\ .05 & .2 & .3 & .35 & .1 & 0 \\ .05 & .2 & .3 & .35 & .1 & 0 \\ .01 & .1 & .1 & .1 & .39 & .3 \\ .2 & .2 & .2 & .2 & .2 & 0 \end{pmatrix}$$

To play it safe Harry starts the evening off with a class 2 performer. What is the probability that a star is discovered (a class 1 performer) before a riot is encountered? What is the expected number of performers seen before the first riot?

Problem 4 *Harry, the Semipro.*

Our hero, Happy Harry, used to play semipro basketball where he was a defensive specialist, His scoring productivity per game fluctuated between three states: 1(scored 0 or 1 points), 2 (scored between 2 and 5 points), 3 (scored more than 5 points). Inevitably, if Harry scored a lot of points in one game, his jealous teammates refused to pass him the ball in the next game, so his productivity in the next game was nil. The team statistician, Mrs. Doc, upon observing the transitions between states, concluded these transitions could be modeled by a Markov chain with transitions could be modeled by a Markov chain with transition matrix

$$P = \begin{pmatrix} 0 & 1/3 & 2/3 \\ 1/3 & 0 & 2/3 \\ 1 & 0 & 0 \end{pmatrix}$$

(1) What is the long run proportion of games that our hero had high scoring games?

(2) The salary structure in the semipro leagues includes incentives for scoring. Harry was paid \$40/game for a high scoring performance, \$30/game when he scored between 2 and 5 points and only \$20/game when he scored nil. What was the long run earning rate of our hero?

Problem 5 *Markov Carlo Markov Chain.*

Background: Suppose we wish to evaluate $\mathbb{E}[h(X)]$ where X has distribution function π ($i.e.$, $\mathbb{P}[X = i] = \pi_i$). The **Monte Carlo** approach is to generate $X_1, X_2, \ldots, X_n \sim \pi$ and estimate $\mathbb{E}[h(X)] \approx \frac{1}{n}\sum_{i=1}^{n} h(X_i)$. If it is hard to generate a iid sample from π, we may look to generate a sequence from a Markov chain with limiting distribution π. This idea, called **Monte Carlo Markov Chain (MCMC)**, was introduced by Metropolis and Hastings (1953). It has become a fundamental computational methods for the physical and biological sciences. It is also commonly used for Bayesian statistical inference.

Metropolis-Hastings Algotithm:

(i) Choose a transition matrix $Q = [q_{ij}]$

(ii) Set $X_0 = 0$

(iii) For $n = 1, 2, \ldots$

(a) Generate Y_n with $\mathbb{P}[Y_n = j | X_{n-1} = i] = q_{ij}$.

(b) If $X_{n-1} = i$ and $Y_n = j$, set $X_n = \begin{cases} j, & \text{with probability} \min(1, \pi_j q_{ji}/\pi_i q_{ij}) \\ i, & \text{otherwise} \end{cases}$

Here Y_n is called the proposal and we say the proposal is accepted with probability $\min(1, \pi_j q_{ji}/\pi_i q_{ij})$. If the proposal is not accepted, the chain stays in its previous state.

Now show that the following proposition:

Set $P_{ij} = \begin{cases} q_{ij} \min(1, \pi_j q_{ji}/\pi_i q_{ij}), & j \neq i \\ q_{ii} + \sum_{k \neq i} q_{ik}\{1 - \min(1, \pi_k q_{ki}/\pi_i q_{ik})\}, & j = i \end{cases}$.

Then π is a stationary distribution of the Metropolis-Hastings chain $\{X_n\}$. If P_{ij} is irreducible and aperiodic, then π is also the limiting distribution.

Problem 6 *Real example about MCMC.*

Suppose that $X_1, X_2, \ldots, X_n$ is sampled from $N(\theta, 1)$, where we assume that $(\log\theta - \mu)/\sigma$ has a t distribution on r degrees of freedom. Assume that μ, σ, r are known.

(1) Let $\pi(\theta \mid X_1, X_2, \ldots, X_n)$ be the posterior distribution and it is given as follows:

$$\pi(\theta \mid X_1, X_2, \ldots, X_n) = \frac{\mathrm{C}}{\theta}\left[1 + \frac{1}{\mathrm{r}}\left(\frac{\log\theta - \mu}{\sigma}\right)^2\right]^{-(\mathrm{r}+1)/2} \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}(X_i - \theta)^2\right\} I(\theta > 0)$$

where C is a constant. Describe a Metropolis-Hastings algorithm for sampling from this posterior distribution $\pi(\theta \mid X_1, X_2, \ldots, X_n)$ using a normal proposal distribution centered at the current value of the Markov chain and with variance τ^2.

(2) Take $\tau^2 = 1, \mu = 0, \sigma = 5$ and $\mathrm{r} = 4$. For the data set $X_1, X_2, \ldots, X_n$ as attached, implement your Metropolis-Hastings algorithm starting the chain at $\theta_0 = 1$ and running for $30,000$ steps.

(3) Record the acceptance rate of your Metropolis-Hastings algorithm. Then create a trace plot in which you plot the values of θ_i against i and also create a histogram of the θ_i values.