Stochastic Process: Exercise # 1

TA Session

February 14th, 2013

Problem 1 *Solve the following exercises.*

(a) Let $X \sim U([0,1])$ and $Y \sim \exp(\lambda)$ be two independent random variables. Compute the distribution of $Z = X + Y$ and $\mathbb{E}[Z^3]$.

(b) Let X_i be independent exponential random variables with parameters $\lambda_i > 0, i = 1, 2$. Compute $\mathbb{P}(X_1 > u + v | X_1 > u), \mathbb{P}(X_1 < X_2)$ and the distribution of $\min\{X_1, X_2\}$.

Problem 2 *Prove the following exercises.*

(a) Let N denote a nonnegative integer-valued random variable. Show that

$$\mathbb{E}[N] = \sum_{k=1}^{\infty} \mathbb{P}\{N \geq k\} = \sum_{k=0}^{\infty} \mathbb{P}\{N > k\} \tag{1}$$

(b) In general show that if X is nonnegative with distribution F, then

$$\mathbb{E}[X] = \int_0^{\infty} \mathbb{P}(X > x) dx \tag{2}$$

$$\mathbb{E}[X^n] = \int_0^{\infty} n x^{n-1} \mathbb{P}(X > x) dx \tag{3}$$

Problem 3 *Prove the following exercises.*

Let X_n denote a binomial random variable, $X_n \sim \text{Binomial}(n, p_n)$ for $n \geq 1$. If $np_n \to \lambda$ as $n \to \infty$, show that

$$\mathbb{P}\{X_n = i\} \to e^{-\lambda} \lambda^i / i! \qquad \text{as} \qquad n \to \infty \tag{4}$$

Problem 4 *One algorithm to simulate sample from any distribution.*

Let F be a continuous distribution function and let U be a uniformly distributed random variable, $U \sim \text{Uniform}(0, 1)$.

(a) If $X = F^{-1}(U)$, show that X has distribution function F.

(b) Show that $-\log(U)$ is an exponential random variable with mean 1.