Notes on Random Walk.

- Probability of eventual return
- Gambler's ruin

RW: Probab. of an eventual return.

Let $\{S_n\}$ a RW, with $S_0 = 0$. Consider

$$p_0(n) = P(S_n = 0) \quad \text{and} \quad f_0(n) = P(S_1 \neq 0, \dots, S_{n-1} \neq 0, S_n = 0)$$

= Prob. of returning to 0 for the 1st. time in n steps.

Remarks (1) $p_0(n)$ is not a p.m.f. but.

$f_0(n)$ is the p.m.f of T_0 = time until the first return to 0.

(2) $p_0(0) = 1$. $p_0(n)$ is defined for $n \geq 0$

$f_0(n)$ is defined for $\boxed{n \geq 1}$

Next, we show the "power" of generating functions.

Consider $P_0(s) = \sum_{n=0}^{+\infty} p_0(n) s^n$ and $F_0(s) = \sum_{n=1}^{+\infty} f_0(n) s^n$.

Note $F_0(s) = \mathbb{E}\left[s^{T_0}\right]$

Lemma: (a) $P_0(s) = 1 + P_0(s) F_0(s)$

(b) $= \left(1 - 4pq s^2\right)^{-1/2}$

(c) $F_0(s) = 1 - \left(1 - 4pq s^2\right)^{1/2}$

Proof: (a) if $n \geq 1$ then

$$p_0(n) = \sum_{k=1}^{n} P(S_1 \neq 0, \dots, S_{k-1} \neq 0, S_k = 0, S_n = 0)$$

$$= \sum_{k=1}^{n} P(S_1 \neq 0, \dots, S_{k-1} \neq 0, S_k = 0)\, P(S_n = 0 \mid S_1 \neq 0, \dots, S_{k-1} \neq 0, S_k = $$

$$(\text{independence}) = \sum_{k=1}^{n} f_0(k)\, P(S_n = 0 \mid S_k = 0) = \sum_{k=1}^{n} f_0(k)\, P(S_{n-k} = 0 \mid S_0 = $$

temporal homogeneity

$$= \sum_{k=1}^{n} p_0(n-k)\, f_0(k)$$

this type of reasoning is very important in this course

So

$$\sum_{k=1}^{+\infty} p_0(n)\, s^n = \sum_{n=1}^{+\infty} \left(\sum_{k=1}^{n} p_0(n-k)\, f_0(k) \right) s^n$$

$$\| \qquad\qquad \|$$

$$P_0(s) - p_0(0)s^0 \qquad \sum_{k=1}^{+\infty} \sum_{n=k}^{+\infty} p_0(n-k)\, f_0(k)\, s^k s^{n-k}$$

$$\| \qquad\qquad \|$$

$$P_0(s) - 1 = P_0(s)\, F_0(s) \quad \blacksquare$$

(b) We know $p_0(n) = P(S_n = 0) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \binom{n}{n/2} (pq)^{n/2} & \text{if } n \text{ is even} \end{cases}$

So

$$P_0(s) = \sum_{n=0}^{+\infty} p_0(2n)\, s^{2n} = \sum_{n=0}^{+\infty} \binom{2n}{n} (pq s^2)^n$$

$$= \left(1 - 4pqs^2\right)^{-1/2} \quad \blacksquare$$

(c) from (a), $F_0(s) = 1 - \frac{1}{P_0(s)}$ ∎

<u>Theorem</u> The probability of a eventual return is $1-|p-q|$

And, if eventual return is certain, then the expected time is $+\infty$

Proof: the probab. of a eventual return is

$$\sum_{k=1}^{+\infty} f_0(n) = F_0(1) = 1-(1-4pq)^{1/2}$$

Note now $1-4pq = (p-q)^2$

By the other hand, the expected time of a eventual return is

$$E(T_0) = F_0'(1) = +\infty$$ ∎

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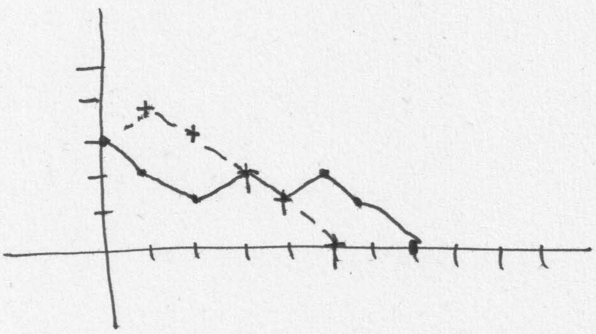
$\boxed{\text{Gambler's Ruin}}$ Consider a game in which you win 1 or lose 1 per turn, with probability p and q. Suppose that you quit playing if your fortune reaches N. and your initial fortune is $k \le N$. Of course, if your fortune reaches 0, your opponent makes you stop.

Two possible paths of the process. with $N=5$ and $k=3$



Let τ be the time when you stop playing

$$\tau = \min\{n \geq 0 : S_n = 0 \text{ or } S_n = N\}$$

(Given $S_0 = k$!) Consider de following probabilities

$$u_k = P(S_\tau = 0 \mid S_0 = 1) = \text{probability of ruin}$$

Conditioning on the first turn and using the law of total probability we get

$$u_k = P(S_\tau = 0 \mid S_1 = k+1)\, p + P(S_\tau = 0 \mid S_1 = k-1)\, q$$

$$= p\, u_{k+1} + q\, u_{k-1}$$

this type of reasoning is very important

$$\Longrightarrow \quad u_{k+1} - u_k = \left(\frac{q}{p}\right)(u_k - u_{k-1}) \quad \text{(check it!)}.$$

Iterating, $u_{k+1} - u_k = \left(\frac{q}{p}\right)^k (u_1 - u_0) = \left(\frac{q}{p}\right)^k (u_1 - 1).$ (1)

Note $u_0 = 1$ and $u_N = 0$!

Now some elementary computes.

Call $r_k = \sum_{n=0}^{k} \left(\frac{q}{p}\right)^n = \begin{cases} k+1 & \text{if } p = q = 1/2 \\ \dfrac{1-(q/p)^{k+1}}{1-(q/p)} & \text{if } p \neq q. \end{cases}$

Then

$$u_{k+1} - 1 = \sum_{n=0}^{k} (u_{n+1} - u_n) \quad (\text{telescopic sum}).$$

$$= \sum_{n=0}^{k} \left(\frac{q}{p}\right)^n (u_1 - 1) \quad (\text{eq. } (1))$$

$$= r_k (u_1 - 1)$$

$$\Longrightarrow u_{k+1} = 1 + r_k(u_1 - 1) \quad \text{in particular}$$

$$u_N = 0 = 1 - r_{N-1}(u_1 - 1) \Longrightarrow u_1 - 1 = -\frac{1}{r_{N-1}}$$

So $u_k = 1 - \dfrac{r_{k-1}}{r_{N-1}} = \begin{cases} \dfrac{N-k}{N} & \text{if } p = 1/2 \\ \dfrac{(q/p)^k - (q/p)^N}{1-(q/p)^N} & \text{if } p \neq 1/2 \end{cases}$