Home Work 4: Martingales and Bowrninan Motion

Stochastic Processes

Deadline April 16, 2013

1. If T_1 and T_2 are stopping times, prove that $T_1 + T_2$ and $\min(T_1, T_2)$ are also stopping times. (2/15)

2. Consider the simple random walk $S_n = S_0 + X_1 + \cdots + X_n$, with $\{X_i\}$ i.i.d random variables with $P(X_i = 1) = P(X_i = -1) = 1/2$. Use Teorema 1, from the slides on martingales, to prove $S_n^2 - n$ is a martingale. (2/15)

3. Let $X_1, X_2, \ldots$ be independent, non negative random variables, with $E[X_i] = 1$. Prove that $M_n = M_0 X_1 \ldots X_n$ is a martinagle. (1/15)

4. Use the optional stopping time theorem to prove the following claim: Let $X_1, X_2, \ldots$ be independent, with $E[X_i] = \mu$ and $Var X_i] = \sigma^2$ adn S_n the random walk $S_n = X_1 + \cdots + X_n$. Let T be a stopping time (we suppose $E[T] < \infty$). Then, $E[S_T^2] = \sigma^2 E[T]$. (2/15)

5. Consider the simple random walk ofn problem 2, with $S_0 = 0$. Use the fact that the mean time for the random walk to first reach $-a < o$ or $b > 0$ is ab and the invariance principle, in th esame way that we did in class, to show that $E[T] = ab$, where

$$T = \min\{t > 0; B(t) = -a \quad \text{or} \quad B(t) = b\}$$

   (3/15)

6. Let $M(t) = \max_{0 \le u \le t} B(u)$. Use the reflection principle to obtain

$$P(M(t) \ge z, B(t) \le x) = P(B(t) \ge 2z - x)$$

   (3/15)

7. Let U_t be a standard Brownian bridge ($U_t = B_t - tB_1$ for $0 \le t \le 1$).
   - Show that $Cov(U_s, U_t) = s(1 - t)$ for $0 \le s \le t \le 1$.
   - Show that $Y_t = (1 + t)U_{t/(1+t)}$ is a BM on $[0, \infty)$. (2/15)