Home Work 2: Markov Chains

Stochastic Processes

Deadline March 19, 2013

1. Prove that the three definitions of the Poisson Process discussed in the classroom are equivalent. (3 points)

2. For each n, let $X_{n,1}, \dots, X_{n,n}$ be independent random variables with

$$P(X_{n,m} = 1) = p_{n,m} \quad \text{and} \quad P(X_{n,m} = 0) = 1 - p_{n,m}.$$

   Let $S_n = X_{n,1} + \cdots + X_{n,n}$ and $\lambda_n = E[S_n]$. Assume

$$\lim_{n\to\infty} \max\{p_{n,m} : 1 \leq m \leq n\} = 0 \quad \text{and} \quad 0 < \lim_{n\to\infty} \lambda_n = \lambda < \infty.$$

   Prove, for any $k \geq 0$,

$$\lim_{n\to\infty} P(S_n = k) = P(Poisson(\lambda) = k)$$

   The above result generalizes the Poisson approximation that you discussed with Professor Alonso. This is

$$Binomial(n, \lambda_n/n) \to Poisson(\lambda), \quad \text{in distribution.}$$

   Can you explain in which sense? (2 points)

3. Given a Poisson process of red arrivals with rate λ and an independent Poisson process of blue arrivals with rate μ, what is the probability that we will get m arrivals before a total of n blue ones?

4. Let $N(t)$ be a nonhomogeneous Poisson process with arrival rate $\lambda(t)$, and let $\Lambda(t) = \int_0^t \lambda(s)ds$. (2 points) Verify

$$P(N(s) = k|N(t) = n) = \frac{n!}{k!(n-k)!} \left(\frac{\Lambda(s)}{\Lambda(t)}\right)^k \left(1 - \frac{\Lambda(s)}{\Lambda(t)}\right)^{n-k}$$

   for any $s < t$ and $k = 0, 1 \dots, n$.

5. Consider the classical Collective Risk Model. Assume that the claims are distributed as a $\Gamma(2, 1)$. Compute the probability of ruin. (2 points)

6. Define the *current life* process by $L(t) = t - T_{N(t)}$. Make some typical paths to show the behavior of the process. Use the regenerative approach to compute the asymptotic distribution of $L(t)$ when $t \to \infty$ and interpret the result. (2 points)

7. Choose and solve one problem, between 7.18 and 7.28, of the Durrett's book (first edition). (2 points)