Home Work 2: Markov Chains

Stochastic Processes

Deadline March 5, 2013

1. Prove the following two claims:
   - if $\rho_{ij} > 0$ and i recurrent then $\rho_{ji} = 1$.
   - if $\rho_{ij} > 0$ and $\rho_{ji} < 1$ then i is transient.
2. Prove the following two claims:
   - if $p(i,i) > 0$ then i is aperiodic
   - if i and j intercommunicate then they have the same period.
3. Suppose you sell a product that is demanded only 0, 1, 2, 3 or 4 units per day, with probabilities 0.10, 0.20, 0.30, 0.25 and 0.15, respectively. Each night, when you close your store you order $4 - s$ units when the number of units on hand is s or below. The new units will be available for the beginning of the next day. Let X_n be the number of units at the end of day n.
   - Explain why $\{X_n\}$ is a MC and write the transition matrix for $s = 0, 1, 2, 3$.
   - Starting with $X_0 = 4$, which is the probability to have no units at the end of the 10th day? Consider all the *inventary policy* $s = 0, 1, 2$, and 3.
   - Why there exists an unique stationary distribution for any transition matrix? Compute these distributions.
   - Suppose also that each sale of the product produces a profit of 10 euros, but keep an unsold unit overnight has a cost equals to 1 euro and it is no possible to sell more than 4 units in a day. Which are the long run profits and costs under each *inventary policy*? Which is the best chose of s?
4. A graph (V, A) is described by the set of vertices V and the adjacency matrix $A(u, v)$, which is equals to 1 if there is an edge from u to v and 0 otherwise. The degree of a vertex u is equal to the number of neighbors it has. Namely,

   $$d(u) = \sum_{v \in V} A(u, v).$$

   The Random Walk on (V, A) is the MC with transition matrix

   $$p(u, v) = \frac{A(u, v)}{d(u)}.$$

   Suppose V is finite and (V, A) is connected (there is a path between any pair of vertices).

- Prove that if $A(u,v) = A(v,u)$ and $A(u,u) = 1$ then there is a unique stationary distribution.
- Compute the stationary distribution and prove that it satisfies a detailed balance condition.

5. Consider the birth-death MC $\{X_n\}$ with transition probabilities

$$P\left[X_{n+1} = j | X_n = i\right] = \begin{cases} p_i = \dfrac{N-i}{N}\dfrac{i}{N-1}, & \text{if } |j-i| = 1 \\ 1 - 2p_i, & \text{if } j = i \\ 0, & \text{if } |j-i| > 1 \end{cases}$$

Note that $0 < X_0 < N$ implies $0 \le X_n \le N$ and that 0 and N are absorbing states. This chain has been used as evolutionary model of finite populations: Suppose that there are N individuals of two types. One individual is randomly selected to die and other is selected to be replaced by two individuals of its type.

- Classify all the states $\{0, 1 \ldots, N\}$.
- Verify that $\pi_\alpha = (\alpha, 0, \ldots, 0, 1-\alpha)$, for any $0 \le \alpha \le 1$, is a stationary distribution (is this information useful?).
- Assume $N = 20$, $X_0 = k$, $1 \le k \le 19$, and compute the probability of visiting 0 before N. What is the expected time until the domination of one type? Hint: use MatLab.