Continuous time Markov Chains

Stochastic Processes

UC3M

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Definition

To extend the Markov property to continuous-time we require defining the conditional probability given we know the process in a continuous range $[0, s]$. That is, given events like $\{X_t, 0 \leq t \leq s\}$.

Def. We say the stochastic process $\{X_t, t \geq 0\}$ taking values in a state space S is a Continuous time Markov Chain if for any sequence of times $0 \leq s_0 < s_1 < \dots s_n < s$ and any set of states $i_0, \dots, i_n, i, j \in S$ the following property holds

$$P(X_{t+s} = j | X_s = i, X_{s_n} = i_n, \dots, X_{s_0} = i_0) = P(X_{t+s} = j | X_s = i)$$

Example. Let $\{N(t), t \geq 0\}$ be the Poisson process with constant rate λ and Y_n a discrete time Markov chain with transition matrix P. Then, the process defined by $X_t = Y_{N(t)}$ is a continuous time MC. Moreover, **the transition matrix** of this chain is

$$P(X_t = j | X_0 = i) = \sum_{n \geq 0} e^{-\lambda t} \frac{(\lambda t)^n}{n!} p^n(i, j)$$

Transition probabilities

As in the discrete case, we will assume the chain is homogenous, i.e.

$$P(X_{s+t} = j | X_s = i) = P(X_t = j | X_0 = i)$$

Def. Let

$$p_t(i,j) = P(X_t = j | X_0 = i)$$

the the transition probability at time t. The matrix $P_t = \{p_t(i,j)\}_{i,j \in S}$ is called the transition matrix at time t

As in the discrete case, the transition probabilitites satify the Chapman-Kolmogorov equations,

$$p_{s+t}(i,j) = \sum_k p_s(i,k) p_t(k,j) \quad i,j \in S$$

or in matrix notation

$$P_{s+t} = P_s P_t$$

Generator of a CT MC

Def. we define transition rate from i to j, $i \neq j$, that we will denote by $q(i,j)$, to

$$q(i,j) = \lim_{t \to 0^+} \frac{p_t(i,j)}{t}$$

This value is interpreted as the intensity with which the chain jumps from i to j. In particular, the rate at which the chain leaves the state i will be denoted by λ_i:

$$\lambda_i = \sum_{j \neq i} q(i,j)$$

Def. We define the Generator of the continuous time Markov chain to the matrix $G = \{g(i,j)\}_{i,j \in S}$, with

$$g(i,j) = \begin{cases} q(i,j) & \text{if } i \neq j \\ -\lambda_i & \text{if } i = j \end{cases}$$

Construction of a continuous time MC

We will say that a state i is

1. **absorbing** if $\lambda_i = 0$,
2. **stable** if $0 < \lambda_i < \infty$ and

If $\lambda_i = \infty$ then the chain will leave immediately from state i, so we will always suppose that $\lambda_i < \infty$.

If $\lambda_i > 0$ define the transition probabilities

$$r(i,j) = q(i,j)/\lambda_i$$

- If $X(s) = i$ and i is absorbing, the the chain will remain in i for ever.
- Otherwise, if the state is stable then the chain will be in i a random time exponentially distributed, with parameter λ_i, and then it will *jump* according to the transition probability $r(i,j)$. Iterating this procedure we can **simulate** lthe continuous time MC.

Kolmogorov's equations

We have discussed how, from the transition rates we can construct the continuous time MC. Since the chain is determined by the transition probabilities, then we must obtain these values also from the transition rates.

The formalization of the above comment is expressed trough the following theorem:

Theorem 1. Assume $\lambda_i < \infty$ for any i in the state space. Then the transition probabilities are differentiable function in t and, for any pair i, j of states we have

$$\frac{dp_t}{dt}(i,j) = \sum_{k \neq i} q(i,k)p_t(k,j) - \lambda_i p_t(i,j) \quad \textbf{Backward equation}$$

$$\frac{dp_t}{dt}(i,j) = \sum_{k \neq j} q(k,j)p_t(i,k) - \lambda_j p_t(i,j) \quad \textbf{Forward equation}$$

Stationary distribution and detailed balance condition

Similar than the definition for the discrete case, π is said to be a stationary distribution if $\pi p_t = \pi$. Or, equivalently,

Def. Let X_t be a continuous time MC with Generator G. Then π is a stationary distribution if it satisfies the equation

$$\pi G = 0$$

Or, equivalently, for each $j \in S$ the following equations hold

$$\sum_{i \neq j} \pi(i) q(i,j) = \lambda_j \pi(j)$$

Def. We say π satisfies a detailed balance condition for the continuos time MC with generator G if for each $i \neq j$

$$\pi(i) g(i,j) = \pi(j) g(j,i)$$

Theorem If π satisfies a detailed balance condition then π is a stationary distribution.

Long-run behavior

Def. A continuous time MC X_t is irreducible if for each $i, j \in S$ there exists a finite sequence of states $k_1 = i, k_2, \ldots, k_{n-1}, k_n = j$ such that $g(k_m, k_{m+1}) > 0$ for $1 \le m < n$.

Theorem. Assume X_t is irreducible and it has stationary distribution π. Then,

$$\lim_{t\to\infty} p_t(i,j) = \pi(j)$$

If $H : S \to \mathbb{R}$ is a function such that $\sum_j |H(j)|\pi(j) < \infty$, then

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t H(X_s)ds = E_\pi(H)$$

with

$$E_\pi(H) = \sum_j H(j)\pi(j)$$

Birth-death Markov chains

Def. A birth-death Markov chains is a continuous time MC with state space $S = \{0, 1, \dots N\}$, $N \leq +\infty$, and generator

$$G = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & 0 & \cdots \\ \mu_1 & -(\mu_1 + \lambda_1) & \lambda_1 & 0 & 0 & \cdots \\ 0 & \mu_2 & -(\mu_2 + \lambda_2) & \lambda_2 & 0 & \cdots \\ \vdots & \vdots & \cdots & \cdots & \cdots & \ddots \end{bmatrix}$$

$\lambda_n \geq 0$ represents the rate of birth $\mu_n \geq 0$ the rate of death when there are n individuals in the population.

The Poisson process is a particular case of birth-death Markov chain with

$$\lambda_n = \lambda \quad n \geq 0$$
$$\mu_n = 0 \quad n \geq 1$$

Stationary distribution of a birth-death Markov chain

π satisfies a **detailed balance condition** for a birth-death MC if

$$\pi(n-1)\lambda_{n-1} = \pi(n)\mu_n, \text{ for } n \geq 1$$

Theorem. Assume $\lambda_n > 0$ for each $n \geq 0$ and $\mu_n > 0$ for each $n \geq 1$. Then, if π satisfies a detailed balance condition for the corresponding birth-death Markov chain then

$$\pi(n) = \frac{\lambda_{n-1}\lambda_{n-2}\cdots\lambda_0}{\mu_n\ \mu_{n-1}\cdots\mu_1}\pi(0), \text{ for } n \geq 1$$

If there exists a stationary distribution for the birth-death Markov chain then

$$\pi(0) = \left(1 + \sum_{n=1}^{N} \frac{\lambda_{n-1}\lambda_{n-2}\cdots\lambda_0}{\mu_n\ \mu_{n-1}\cdots\mu_1}\right)^{-1}$$

Some examples of queues

- **M/M/1** corresponds to a birth-death Markov chain with $\lambda_n = \lambda > 0$ and $\mu_n = \mu > 0$. Moreover, we will assume $\lambda < \mu$. The stationary distribution corresponds to a Geometric random variable $\frac{\lambda}{\mu}$, i.e.

$$\pi(n) = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n \textit{ for } n \geq 0$$

- **M/M/∞** corresponds to a birth-death Markov chain with $\lambda_n = \lambda > 0$ and $\mu_n = n\mu > 0$. The stationary distribution is Poisson with parameter $\frac{\lambda}{\mu}$, i.e.

$$\pi(n) = \frac{1}{n!} e^{-\frac{\lambda}{\mu}} \left(\frac{\lambda}{\mu}\right)^n \textit{ for } n \geq 0$$

Exist distribution and hitting times

The results on exit distribution and hitting times from discrete time MC can be generalized to continuous time. The approach is focussed in the embedded jump chain with transition probability $r(i,j)$.

Example: the $M/M/1$ queue.

In this case, $q(i,i+1)=\lambda$ and $q(i,i-1)=\mu$. So, the embedded MC has transition matrix, $r(0,1)=1$ and

$$r(i,i+1)=\frac{\lambda}{\lambda+\mu},\quad r(i,i-1)=\frac{\mu}{\lambda+\mu},\quad \text{for}\quad i\geq 1$$

This above corresponds to a reflected Random walk with that is positive. Therefore, we can characterize the queue, which is

- positive recurrent if $\lambda<\mu$
- null recurrent if $\lambda=\mu$
- transient if $\lambda>\mu$