Some useful distributions

Discrete distributions

1. **Binomial**
   $X \sim \mathcal{BI}(n, \theta)$ if
   $$P(X = x) = \binom{n}{x} \theta^x (1-\theta)^{n-x} \quad x = 0, 1, \ldots, n.$$
   $E[X] = n\theta$ and $V[X] = n\theta(1-\theta)$.

2. **Discrete uniform**
   $X \sim \mathcal{DU}[\theta_1, \theta_2]$ if
   $$P(X = x) = \frac{1}{\theta_2 - \theta_1 + 1} \quad x = \theta_1, \theta_1 + 1, \ldots, \theta_2.$$
   $E[X] = \frac{\theta_1 + \theta_2}{2}$ and $V[X] = \frac{(\theta_2 - \theta_1)(\theta_2 - \theta_1 + 2)}{12}$.

3. **Geometric**
   $X \sim \mathcal{GE}(\theta)$ if
   $$P(X = x) = \theta(1-\theta)^x \quad x = 0, 1, 2, \ldots$$
   $E[X] = \frac{1-\theta}{\theta}$ and $V[X] = \frac{1-\theta}{\theta^2}$.

4. **Hypergeometric**
   $X \sim \mathcal{H}(N, R, n)$ if
   $$P(X = x) = \frac{\binom{R}{x}\binom{N-R}{n-x}}{\binom{N}{n}} \quad x = 0, 1, \ldots, n$$
   $E[X] = n\frac{R}{N}$ and $V[X] = n\frac{R}{N}\left(1 - \frac{R}{N}\right)\frac{N-n}{N-1}$.

5. **Negative Binomial**
   $X \sim \mathcal{NB}(r, \theta)$ if
   $$P(X = x) = \binom{x+r-1}{x} \theta^r (1-\theta)^x \quad x = 0, 1, 2, \ldots$$
   $E[X] = r\frac{1-\theta}{\theta}$ and $V[X] = r\frac{(1-\theta)}{\theta^2}$.

6. **Poisson**

   $X \sim \mathcal{P}(\theta)$ if

$$P(X = x) = \frac{\theta^x e^{-\theta}}{x!} \quad x = 0, 1, 2, \ldots$$

   $E[X] = \theta = V[X]$.

Continuous distributions

1. **Behrens Fisher**

   $X \sim \mathcal{BF}(\nu_1, \nu_2, \theta)$ if $X = T_1 \cos\theta - T_2 \sin\theta$ and $T_i \sim \mathcal{T}(\nu_i, 0, 1)$.

   $E[X] = 0$ for $\nu_1, \nu_2 > 1$ and $V[X] = \frac{\nu_1 \sin^2\theta}{\nu_1 - 2} + \frac{\nu_2 \cos^2\theta}{\nu_2 - 2}$ for $\nu_1, \nu_2 > 2$.

2. **Beta**

   $X| \sim \mathcal{B}(\alpha, \beta)$ if

$$f(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1 - x)^{\beta-1} \quad 0 < x < 1,$$

   where $B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.

   $E[X] = \frac{\alpha}{\alpha+\beta}$ and $V[X] = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$.

3. **Cauchy**

   $X \sim \mathcal{C}(\mu, \sigma^2)$ if

$$f(x) = \frac{1}{\pi} \frac{\sigma}{\sigma^2 + (x - \mu)^2} \quad \forall\, x.$$

   The moments of this distribution do not exist.

4. **Chi-squared**

   $X \sim \chi^2_\nu$ if $X \sim \mathcal{G}(\nu/2, 1/2)$.

$$f(x) = \frac{2^{-\nu/2}}{\Gamma(\nu/2)} x^{\nu/2-1} e^{-x/2} \quad x > 0.$$

   $E[X] = \nu$ and $V[X] = 2\nu$.

5. **Exponential**

   $X \sim \mathcal{E}(\theta)$ if

$$f(x) = \theta e^{-\theta x} \quad x > 0.$$

   $E[X] = \frac{1}{\theta}$ y $V[X] = \frac{1}{\theta^2}$.

6. **Fisher's F**
   $X \sim \mathcal{F}(\alpha, \beta)$ if
   $$f(x) = \frac{1}{B(\alpha/2, \beta/2)} \left(\frac{\alpha}{\beta}\right)^{\alpha/2} x^{\alpha/2-1} \left(1 + \frac{\alpha}{\beta} x\right)^{-\frac{\alpha+\beta}{2}} \quad \forall\, x.$$
   $E[X] = \frac{\alpha}{\beta-2}$ if $\beta > 2$ and $V[X] = \frac{2\beta^2(\alpha+\beta-2)}{\alpha(\beta-2)^2(\beta-4)}$ if $\beta > 4$.
7. **Gamma**
   $X| \sim \mathcal{G}(\alpha, \beta)$ if
   $$f(x) = \frac{\beta}{\Gamma(\alpha)} (\beta x)^{\alpha-1} e^{-\beta x} \quad x > 0.$$
   $E[X] = \frac{\alpha}{\beta}$ and $V[X] = \frac{\alpha}{\beta^2}$.
8. **Inverse Chi-squared**
   $X \sim \mathcal{I}\chi^2_\nu$ if $1/X \sim \chi^2_\nu$. $E[X] = \frac{1}{\nu-2}$ if $\nu > 2$ and $V[X] = \frac{2}{(\nu-2)^2(\nu-4)}$ if $\nu > 4$.
9. **Inverse Gamma**
   $X \sim \mathcal{IG}(\alpha, \beta)$ if $X^{-1} \sim \mathcal{G}(\alpha, \beta)$.
   $$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{-(\alpha+1)} e^{-\beta/x} \quad x > 0.$$
   $E[X] = \frac{\beta}{\alpha-1}$ if $\alpha > 1$ and $V[X] = \frac{\beta^2}{(\alpha-1)^2(\alpha-2)}$ if $\alpha > 2$.
10. **Normal**
    $X \sim N(\mu, \sigma^2)$ if
    $$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} \quad \forall x.$$
    $E[X] = \mu$ and $V[X] = \sigma^2$.
11. **Pareto**
    $X \sim \mathcal{PA}(\alpha, \beta)$ if
    $$f(x) = \beta\alpha^\beta x^{-\beta-1} \quad x > \alpha.$$
    $E[X] = \frac{\alpha\beta}{\beta-1}$ if $\beta > 1$ and $V[X] = \frac{\alpha^2\beta}{(\beta-1)(\beta-2)}$ if $\beta > 2$.
12. **Student's t**
    $X \sim \mathcal{T}(\nu, \mu, \sigma^2)$ if
    $$f(x) = \frac{\Gamma((\nu+1)/2)}{\Gamma(\nu/2)\sqrt{\nu\pi}\sigma} \left(1 + \frac{1}{\nu}\left(\frac{x-\mu}{\sigma}\right)^2\right)^{-(\nu+1)/2} \quad \forall\, x.$$
    $E[X] = \mu$ if $\nu > 1$ and $V[X] = \frac{\nu}{\nu-2}\sigma^2$ if $\nu > 2$.
    The $\mathcal{T}(\nu, 0, 1)$ distribution is the standard Student's t distribution.

13. **Uniform**
$X \sim \mathcal{U}(0, \theta)$ if

$$f(x) = \frac{1}{\theta} \quad 0 < x < \theta.$$

$E[X] = \frac{\theta}{2}$ and $V[X] = \frac{\theta^2}{12}$.

Multivariate distributions

1. **Dirichlet**
$\mathbf{X} = (X_1, \ldots, X_p)^T \sim \mathcal{D}(\boldsymbol{\theta})$ if

$$f(\mathbf{x}) = \frac{\Gamma\left(\sum_{i=1}^p \theta_i\right)}{\Gamma(\theta_1)\cdots\Gamma(\theta_p)} \prod_{i=1}^p x_i^{\theta_1 - 1} \quad 0 < x_1 < 1, \sum_{i=1}^p x_i = 1$$

$E[X_j] = \frac{\theta_j}{\sum_{i=1}^p \theta_i}$ and $V[X_j] = \frac{\theta_j(1-\theta_j)}{(\sum_{i=1}^p \theta_i)^2 (\sum_{i=1}^p \theta_i + 1)}$

2. **Multinomial**
$\mathbf{X} = (X_1, \ldots, X_p)^T \sim \mathcal{MN}(n, \boldsymbol{\theta})$ if

$$P(\mathbf{X} = \mathbf{x}) = \frac{n!}{\prod_{i=1}^p x_i!} \prod_{i=1}^p \theta_i^{x_i} \quad \textstyle\sum_{i=1}^p x_i = n \text{ and } 0 < \theta_i < 1\ \forall i$$

$E[X_j] = n\theta_j$ and $V[X_j] = n\theta_j(1 - \theta_j)$.

3. **Multivariate normal**
$\mathbf{X} = (X_1, \ldots, X_p)^T \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ if

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{p/2} |\boldsymbol{\Sigma}|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu})\right)$$

$E[\mathbf{X}] = \boldsymbol{\mu}$ and $V[\mathbf{X}] = \boldsymbol{\Sigma}$.

4. **Wishart**
$\mathbf{V} \sim \mathcal{W}(\nu, \boldsymbol{\Sigma})$ if

$$f(\mathbf{V}) = \left(2^{\nu k/2} \pi^{k(k-1)/4} \prod_{i=1}^k \Gamma\left(\frac{\nu + 1 - i}{2}\right)\right)^{-1} |V|^{-\nu/2} |\boldsymbol{\Sigma}|^{(\nu - k - 1)/2} \exp\left(-\frac{1}{2} tr(\boldsymbol{\Sigma}^{-1} \mathbf{V})\right)$$

where $k = \dim \mathbf{V}$.
$E[\mathbf{V}] = \nu \boldsymbol{\Sigma}$.

5. **Inverse Wishart**
$\mathbf{V} \sim \mathcal{WI}(\nu, \boldsymbol{\Sigma}^{-1})$ if $\mathbf{V} \sim \mathcal{W}(\nu, \boldsymbol{\Sigma}^{-1})$.
$E[\mathbf{V}] = (\nu - k - 1)^{-1} \boldsymbol{\Sigma}$.