Chapter 3

Laws of large numbers

The laws of large numbers are a collection of theorems that establish the convergence, in some of the ways already studied, of sequences of the type $\{n^{-1}\sum_{i=1}^{n} X_i - a_n\}$, where a_n is a constant generally given by $a_n = n^{-1}\sum_{i=1}^{n} E(X_i)$. These theorems are classified as weak or strong laws, depending on whether the convergence is in probability or almost surely.

3.1 Weak laws of large numbers

Theorem 3.1 (Tchebychev's Theorem)
Let $\{X_n\}_{n\in I\!N}$ be a sequence of independent r.v.s (not necessarily identically distributed) such that $V(X_n) \leq M < \infty$, $\forall n \in I\!N$. Then,

$$\frac{1}{n}\sum_{i=1}^{n} X_i - \frac{1}{n}\sum_{i=1}^{n} E(X_i) \xrightarrow{P} 0.$$

Corolary 3.1 (Tchebychev's Theorem for r.v.s with equal mean)
In the conditions of Theorem 3.1, if $E(X_n) = \mu$, $\forall n \in I\!N$, then

$$\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{P} \mu.$$

Thus, the sample mean converges weakly to the population mean. Historically, the next corollary was the first law of large numbers.

Corolary 3.2 (Bernouilli's Theorem)
Let $\{X_n\}_{n\in I\!N}$ be a sequence of i.i.d. r.v.s distributed as Bern(p). Then,

$$\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{P} p.$$

The next theorem does not require the existence of the variances, but in turn requires the r.v.s to be identically distributed.

Theorem 3.2 (Khintchine's weak law of large numbers)
Let $\{X_n\}_{n\in I\!N}$ be a sequence of i.i.d. r.v.s with mean $E(X_n) = \mu \in (-\infty, \infty)$. Then,

$$\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{P} \mu.$$

3.2 Strong law of large numbers

Lemma 3.1 (Kolmogorov's bound)
Let $\{X_n\}_{n\in I\!N}$ be a sequence of independent r.v.s with mean $E(X_n) = \mu_n$ and $V(X_n) = \sigma_n^2$, both finite. Let $S_n = \sum_{i=1}^n X_i$ and $c_n^2 = \sum_{i=1}^n \sigma_i^2$. Then, it holds that for all $H > 0$,

$$P\left(\bigcup_{k=1}^{n} \{\omega \in \Omega : |S_k(\omega) - E(S_k)| \geq Hc_n\}\right) \leq \frac{1}{H^2}.$$

Theorem 3.3 (Kolmogorov's strong law of large numbers)
Let $\{X_n\}_{n\in I\!N}$ be a sequence of independent r.v.s with mean $E(X_n) =$

μ_n and $V(X_n) = \sigma_n^2$, both finite.

$$\text{If } \sum_{n=1}^{\infty} \frac{\sigma_n^2}{n^2} < \infty, \text{ then } \frac{1}{n}\sum_{i=1}^{n} X_i - \frac{1}{n}\sum_{i=1}^{n} \mu_i \stackrel{a.s.}{\to} 0.$$

Corolary 3.3 (Borel-Cantelli's Theorem)
Let $\{X_n\}_{n \in I\!N}$ be a sequence of i.i.d. r.v.s distributed as Bern(p). Then,

$$\frac{1}{n}\sum_{i=1}^{n} X_i \stackrel{a.s.}{\to} p.$$

This theorem says that the relative frequency of a dichotomic event goes almost surely to the probability of the event.

Finally, the next strong law does not require anything to the variances but it assumes that the r.v.s are i.i.d.

Theorem 3.4 (Khintchine's strong law of large numbers)
Let $\{X_n\}_{n \in I\!N}$ be a sequence of i.i.d. r.v.s with $E(X_n) = \mu < \infty$. Then

$$\frac{1}{n}\sum_{i=1}^{n} X_i \stackrel{a.s.}{\to} \mu.$$