Introduction to Probability

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Outline

1. Introduction
2. Random phenomena
   - Events, operations with events and their properties
3. Definition of probability and properties
   - Definition of probability
   - Elementary properties
   - Interpretations of probability
4. Conditional probability
   - Independence of events
   - Definition of conditioned probability
5. Bayes Theorem
   - Total probability rule and Bayes Theorem

Introduction

- The Probability Calculus is used to model random phenomena. It endows the conclusions about datasets drawn by statistical inference with mathematical rigor.

- A probability is used to quantify the likelihood, or chance, that an outcome of a random experiment will occur.

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Random phenomena

- An experiment is deterministic if when repeated in the same manner always leads to the same outcome.
- An experiment is random if although it is repeated in the same manner every time, can result in different outcomes. More specifically:
  - The set of all possible outcomes is completely determined before carrying it out.
  - Before we carry it out, we cannot predict its outcome.
  - It can be repeated indefinitely, always under the same conditions (leading to different outcomes).

Events

- The sample space is the set of all possible outcomes of a random experiment, we will denote it by E.
  - Example: Experiment, roll a die, $E=\{1,2,3,4,5,6\}$

- An event is a subset of the sample space (any set of outcomes of the random experiment).
  - An elementary event (singleton) is an element of the sample space.
    - Example: (roll a die), getting a six, $A=\{6\}$
  - A compound event is a set of elementary events.
    - Example: (roll a die), getting an even number $B=\{2,4,6\}$

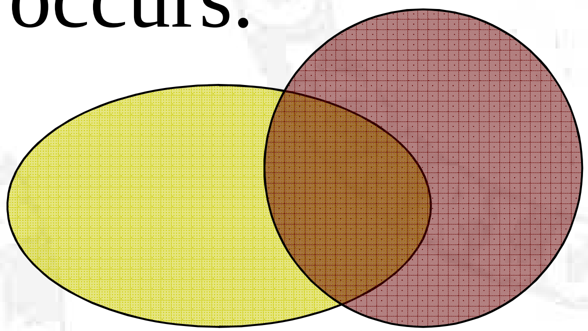
Events

- The certain event is the one that always occurs as an outcome of the experiment, *E*.
  - Example: (roll a die) $E=\{1,2,3,4,5,6\}$

- The null (impossible) event is the one that never occurs, ∅.
  - Example: (roll a die) getting a negative result

Operations with events (sets)

- Union. Given two events A and B, the event $A \cup B$ ocurrs when either of them (or both simultaneously) occurs.

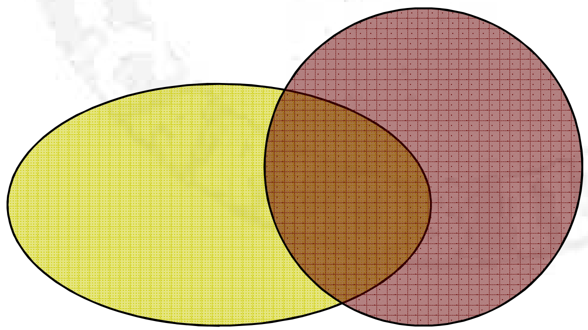


A={☺,☺,☺,☹} ; B={☹,☹,☺,☹}

$A \cup B$={☺,☺,☹,☹,☺,☹}

Operations with events (sets)

- Intersection. Given two events A and B, the event $A \cap B$ (or AB) ocurrs when A and B simultaneously occur.



A={☺,☺,☺,☹} ; B={☹,☹,☺,☹}

$A \cap B$={☺,☹}

Operations with events (sets)

- Complementary event. Given an event *A*, its complementary event A^c occurs when *A* does not.

E={☺,☺,☺} ; A={☺}

A^c={☺,☺}

Operations with events (sets)

- Event (set) difference. Given two events A and B, its difference $A \backslash B$ (or A-B) occurs when A occurs, but B does not.

$$A=\{☺,☺,☺\}\ ;\ B=\{☺,☹\}$$

$$A \backslash B=\{☺,☺\}.$$

- Mutually exclusive events. Two events A and B are mutually exclusive (disjoint) if

$$A \cap B=\emptyset$$

Properties of the operations with events

- Commutative.

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

- Associative.

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

Properties of the operations with events

- Neutral element.
  - Union, null event: $A \cup \emptyset = A$
  - Intersection, certain event: $A \cap E = A$
- Distributive.
  - Union wrt intersection

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

  - Intersection wrt union

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Properties of the operations with events

- Complementation. (exhaustive and mutually exclusive)

$$A \cup A^c = E \;\; ; \;\; A \cap A^c = \emptyset$$

- Idempotence.

$$A \cup A = A \;\; ; \;\; A \cap A = A$$

- Absortion.

$$A \cup E = E \;\; ; \;\; A \cap \emptyset = \emptyset$$

- Simplification. $A \cup (A \cap B) = A = A \cap (A \cup B)$

Properties of the operations with events

- Properties of the complementary event.

$$(A^c)^c = A \;\;;\;\; E^c = \emptyset \;\;;\;\; \emptyset^c = E$$

- DeMorgan's laws.

$$(A \cup B)^c = A^c \cap B^c$$

$$(A \cap B)^c = A^c \cup B^c$$

$$(\cup_{i=1,\infty} A_i)^c = \cap_{i=1,\infty} (A_i)^c$$

$$(\cap_{i=1,\infty} A_i)^c = \cup_{i=1,\infty} (A_i)^c$$

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Definition of probability

Probability is a number assigned to each member of a collection of events from a random experiment that satisfies the following properties:

1. $P(A) \geq 0$;
2. $P(E) = 1$;
3. if $A_1, A_2, \ldots$ are such that $A_i \cap A_j = \varnothing$ when $i \neq j$, then $P(\cup_{i=1,\infty} A_i) = \Sigma_{i=1,\infty} P(A_i)$.

Elementary properties of a probability

- Property 1. $P(A^c) = 1 - P(A)$
- Property 2. $P(\emptyset) = 0$
- Property 3. if $A \subset B$, then $P(A) \leq P(B)$
- Property 4. $\mathrm{P}(A \setminus B) = P(A) - P(A \cap B)$
- Property 5. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Laplace rule (equiprobability)

When a random experiment has a finite number of equally likely possible outcomes, then for any event A

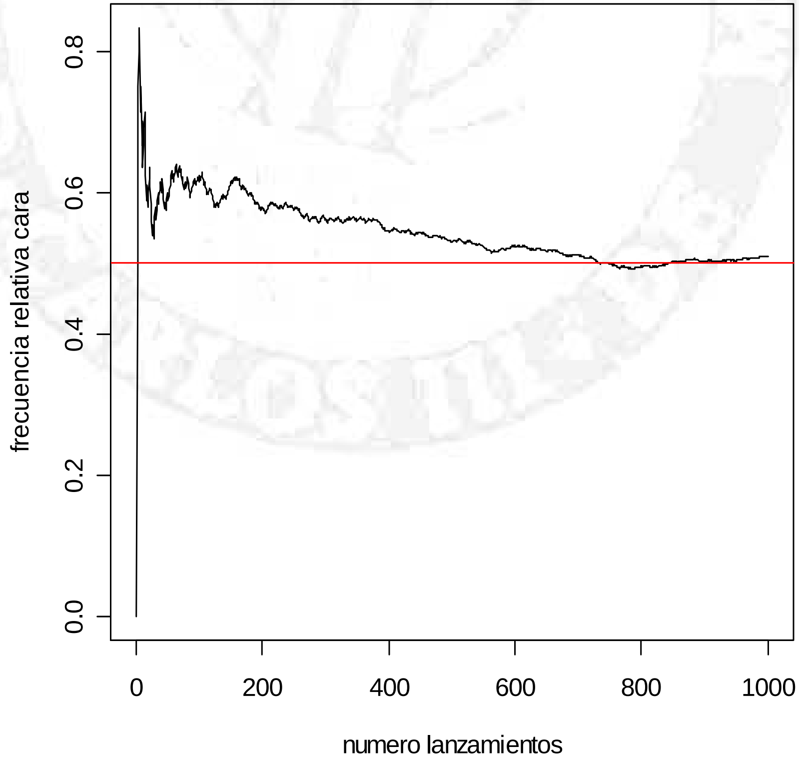
$$P(A) = \frac{\text{number of outcomes favorable to } A}{\text{total number of possible outcomes}}$$

Interpretations of probability

- Law of Large Numbers.

  After a large number of realisations of a random experiment, the relative frequency of any event *A* stabilises around a value

  (PROBABILITY OF *A*)



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Independence of events

Two events A and B are independent if

$$P(A\cap B)=P(A)P(B)$$

$P(A\cap B)$	$P(A\cap B^c)$
$P(A^c\cap B)$	$P(A^c\cap B^c)$

Conditional probability

Given two events A and B with $P(B)>0$, the conditional probability of A given B is the probability that A occurs given that B has occurred,

$$P(A|B)=\frac{P(A\cap B)}{P(B)}$$

When A and B are independent, $P(A|B)=P(A)$.

Conditional probability

It holds:

- $P(A|B) \geq 0$;
- $P(E|B) = 1$;
- if $A_1, A_2, \ldots$ are such that $A_i \cap A_j = \emptyset$ when $i \neq j$, then $P(\cup_{i=1,\infty} A_i|B) = \Sigma_{i=1,\infty} P(A_i|B)$.

  As a consequence, all the properties of a probability, hold for a conditional probability.

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Multiplication rule

Given n events $A_1, A_2, \ldots, A_n$ with $P(A_i)>0$ for $i=1,\ldots,n$. It holds

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1)P(A_2|A_1)\ldots P(A_n|A_1 \cap A_2 \cap \ldots \cap A_{n-1})$$

Further, if the events are independent

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1)P(A_2)\ldots P(A_n)$$

Total probability rule

Given $A_1, A_2,\ldots,A_n$ such that $A_i \cap A_j = \emptyset$ when $i \neq j$ and $\cup_{i=1,n} A_i = E$ (mutually exclusive and exhaustive), the probability of any event B is

$$P(B) = \sum_{i=1}^{n} P(A_i) P(B|A_i)$$

Bayes Theorem

Given $A_1, A_2,\ldots,A_n$ such that $A_i \cap A_j = \emptyset$ when $i \neq j$ and $\cup_{i=1,n} A_i = E$ (mutually exclusive and exhaustive), and B such that $P(B)>0$, it holds

$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(A_i)P(B|A_i)}{\sum_{j=1}^{n} P(A_j)P(B|A_j)}$$

Example (washers)

Suppose that a lot of washers L_1 is large enough that it can be assumed that the sampling is done with replacement. Assume that 60% of the washers exceed the target thickness.

Washers are selected from the lot L_1:

- If 3 washers are selected, what is the probability that the first 2 of them exceed the target thickness, but the last one does not?
- If 3 washers are selected, what is the probability that exactly 2 of them exceed the target thickness?
- What is the minimum number of washers that need to be selected so that the probability that all the washers are thicker than the target is less than 0.10?

Example (washers)

Another big lot of washers, containing twice as many washers as the first one (L_1) is purchased. Out of the washers from this second lot, 40% exceed the target thickness. The two lots are merged together into a new lot, and washers are selected from it.

- If a washer exceeds the target thickness, what is the probability that it comes from the first lot (L_1)?