Extraordinary exam for Probability
Master in Statistics for Data Science
25 June, 2019

P1. (1.5 points) One hundred people line up to board an airplane with one hundred seats. Each has a boarding pass with assigned seat. However, the first person to board has lost his boarding pass and takes a random seat. After that, each person takes the assigned seat if it is unoccupied, and one of unoccupied seats at random otherwise.

a) (0.25 points) What is the probability that the first person to board gets to sit in his assigned seat?

b) (0.5 points) What is the probability that the second person to board gets to sit in his assigned seat?

c) (0.25 points) What is the probability that the third person to board gets to sit in his assigned seat?

d) (0.5 points) What is the probability that the last person to board gets to sit in his assigned seat?

P2. (2 points) The cdf of the shifted (or two-parameter) exponential distribution with origin parameter $\theta \in \mathbb{R}$ (sometimes restricted to $\theta \geq 0$) and scale parameter $\lambda > 0$ is given by

$$F(x) = 1 - e^{-(x-\theta)/\lambda}\,, \quad x > \theta\,,$$

a) (0.25 points) Show that if X is an exponential random variable with parameter $\lambda > 0$ and $\theta \in \mathbb{R}$, then $X + \theta$ is a shifted exponential random variable with parameters θ and λ.

b) (0.25 points) Determine the density mass function of a shifted exponential random variable.

c) (0.25+0.25 points) Determine the mean and variance of a shifted exponential random variable.

d) (0.5 points) The shifted exponential distribution is commonly used in reliability studies to model lifetime data. In this setting, the origin parameter θ can be interpreted as the length of the guarantee period. If the lifetime of a device (in years) is modelled as a shifted exponential random variable with parameters $\theta = 1$ and $\lambda = 0.5$, what lifetime is exceeded by 75% of the devices?

e) (0.5 points) Write a piece of code to simulate 1000 observations of a shifted exponential random variable with origin parameter $\theta = 1$ and scale parameter $\lambda = 0.5$. Use your simulations to approximate the answer to part d).

P3. (3.25 points) The number of items bought by each client visiting a small shop follows a Poisson distribution with mean 3.5 items.

a) (0.25 points) What is the probability of having a client that buys more than 3 items?

b) (0.25 points) What is the probability that, out of all clients visiting the shop in a given day, the first one that buys more than 3 items is the fourth one?

c) (0.25 points) What is the probability that at least 2 of the last 10 clients buy more than 3 items each?

d) (0.5 points) Approximate the probability that the total number of items bought by the last 10 clients is greater than 30.

e) (0.5 points) How many visits are needed so that the probability of selling more than 100 items is at least 0.95?

f) (0.5 points) What is the probability that at least 20 of the last 100 clients visiting the shop bought more than 3 items each?

g) (0.5 points) What is the probability that in a group of 10 clients taken at random, there are exactly 2 clients that bought less than two items and 4 clients that bought two or three items.

h) (0.5 points) A group of 100 clients visiting the shop has been selected in order to study all available data about them. A total number of 42 of those 100 clients bought more than 3 items. If a set of 10 visits is taken at random from the group of 100, what is the probability that more than 5 of them are of clients that bought more than 3 items.

P4. (2.25 points) A statistics class takes two exams X (Exam 1) and Y (Exam 2) where the scores follow a bivariate normal distribution with parameters:

$$\mu_X = 70 \text{ and } \mu_Y = 60 \text{ are the marginal means,}$$

$$\sigma_X = 10 \text{ and } \sigma_Y = 15 \text{ are the marginal standard deviations,}$$

$$\rho = 0.6 \text{ is the correlation coefficient.}$$

Suppose we select a student at random. What is the probability that...

a) (0.25 points) ...the student scores over 75 on Exam 2?

b) (0.5 points) ...the student scores over 75 on Exam 2, given that the student scored $x = 80$ on Exam 1?

c) (0.5 points) ...the sum of his/her Exam 1 and Exam 2 scores is over 150?

d) (0.5 points) ...the student did better on Exam 1 than Exam 2?

e) (0.5 points) Assume the final grade is computed as $0.4X + 0.6Y$. What final grade is exceeded by 65% of the students?

P5. (1 point) During the Middle Ages, coins struck by the Royal Mint in England were evaluated for their metal content on a sample basis, in a ceremony called the Trial of the Pyx. One hundred gold coins were chosen at random from all of the coins made at the Mint every given year, put in the Pyx (a ceremonial box), and weighed. Each of these gold coins was supposed to weigh 128 grains, so the 100 coins in the Pyx should have weighed about 12800 grains. A margin of error of 32 grains was allowed at the trial, but if the actual weight of the coins in the Pyx was less than $12800 - 32 = 12768$ grains, the Master of the Mint was exposed to serious penalties. Assume that the Master of the Mint is honest and manufactures gold coins with a mean weight of 128 grains and standard deviation of 1 grain.

a) (0.5 points) What are the mean and variance of the total weight of the 100 coins?, and the (approximate) distribution of the total weight of the 100 coins?

b) (0.25 points) What is the chance that the Master of Mint survives the Trial of the Pyx?

c) (0.25 points) What margin of error should be fixed so that the probability that a honest Master of Mint survives the Trial of the Pyx is 0.99?