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6. Properties of the expectation

Problem 1. Consider the five-parameter model given by the mixture of two normal distributions. Denote $X_1 \sim \mathrm{N}(\mu_1, \sigma_1)$, $X_2 \sim \mathrm{N}(\mu_2, \sigma_2)$, and X a random variable whose cdf is $F_X = pF_{X_1} + (1-p)F_{X_2}$.

1. Determine the first four moments of X_1 and X_2.
2. Determine the first four moments of X.
3. What are the values of the first four moments of X if $\mu_1 = 2$, $\sigma_1 = 1$, $\mu_2 = 3$, $\sigma_2 = 2$, and $p = 0.4$? Simulate $n = 10000$ observations from that distribution model and check your results (use your DNI or NIU as seed for your simulations).

Problem 2. Claims arrive in an insurance company with regard to a Poisson process, that is, the number of claims received in a time period is a Poisson random variable N. The size of each individual claim is a random variable X_i that models its cost. The X_is are independent and independent of N. Assume that each individual claim X_i follows a log-normal distribution with parameters μ and σ, that is, $\log(X_i) \sim \mathrm{N}(\mu, \sigma)$ and the mean and variance of this distribution model are

$$\mathbb{E}[X_i] = e^{\mu+\sigma^2/2} \quad \text{and} \quad \mathrm{Var}[X_i] = (e^{\sigma^2} - 1)e^{2\mu+\sigma^2}.$$

The total expenditure that the insurance company must carry out to cover the claims is

$$Y = \sum_{i=1}^{N} X_i \,.$$

1. Determine $\mathbb{E}[Y]$ and $\mathrm{Var}[Y]$.
2. What values do $\mathbb{E}[Y]$ and $\mathrm{Var}[Y]$ assume if $\lambda = 100$, $\mu = 6$, and $\sigma = 2.5$?
3. Simulate $n = 10000$ observations of Y in order to check your results (use your DNI or NIU as seed for your simulations).
4. Assume now that the size of each individual claim, X_i, follows a mixture of two log-normal distributions with probability 1/2 each and respective parameters μ_1 and σ_1, and μ_2 and σ_2. Determine $\mathbb{E}[Y]$ and $\mathrm{Var}[Y]$.