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Theory and Methodology

Representation of consistent recursive rules

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Abstract

This paper develops the recursive model for connective rules (as proposed in V. Cutello, E. Molina, J. Montero, Associativeness versus recursiveness, in: Proceedings of the 26th IEEE International Symposium on Multiple-valued Logic, Santiago de Compostela, Spain, 29–31 May, 1996, pp. 154–159; V. Cutello, E. Molina, J. Montero, Binary operators and connective rules, in: M.H. Smith, M.A. Lee, J. Keller, J. Yen (Eds.), Proceedings of NAFIPS 96, North American Fuzzy Information Processing Society, IEEE Press, Piscataway, NJ, 1996, pp. 46–49), where a particular solution in the *Ordered Weighted Averaging (OWA)* context (see V. Cutello, J. Montero, Recursive families of OWA operators, in: P.P. Bonissone (Ed.), Proceedings of the Third IEEE Conference on Fuzzy Systems, IEEE Press, Piscataway, NJ, 1994, pp. 1137–1141; V. Cutello, J. Montero, Recursive connective rules, International Journal of Intelligent Systems, to appear) was translated into a more general framework. In this paper, some families of solutions for the key recursive equation are obtained, based upon the general associativity equation as solved by K. Mak (Coherent continuous systems and the generalized functional equation of associativity, Mathematics of Operations Research 12 (1987) 597–625). A context for the representation of families of binary connectives is given, allowing the characterization of key families of connective rules. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

As presented in previous papers [2–4], aggregation procedures classically have been defined as rules, telling us how to proceed with the information reaching to us; its dimension not being previously fixed. For example, the *maximum rule* applied to real numbers refers to the maximum of any finite subset of numbers; the *mean rule* evaluates the mean for any finite set of real numbers; and the *median rule* evaluates the

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median for arbitrary finite families of numbers. Therefore, a rule is always understood as a consistent family of connectives,

$$\{\phi_n : L^n \rightarrow L\}_{n>1},$$

with each ϕ_n being in charge of the amalgamated evaluation (in terms of the evaluation set L) of any family of n information items (see also [18,19]).

For simplicity, we shall assume here that the evaluation set is $L = [0, 1]$ and by definition we also assume that all connectives in this paper are continuous and non-decreasing in each coordinate.

A standard way to deal with this issue is to restrict our rule models to those explained by means of a unique associative and commutative binary operator

$$\phi_2 : [0, 1]^2 \rightarrow [0, 1]$$

with ϕ_2 being continuous and non-decreasing in each coordinate (see [8,9]) in such a way that the aggregated value of any permutation of $(a_1, \dots, a_n) \in \mathfrak{R}^n$ always takes the same value:

$$\phi_n(a_1, \dots, a_n) = \phi_2(\phi_2(\dots \phi_2(\phi_2(a_1, a_2), a_3), \dots), a_n) = \phi_2(a_1, (\dots, \phi_2(a_{n-2}, \phi_2(a_{n-1}, a_n)) \dots)).$$

Standard fuzzy connectives (t -norms and t -conorms, or even uninorms [23,12]) are associative and commutative binary operators. For example, the *maximum rule* can be obtained by means of a sequential application of the maximum operator of two numbers, no matter the order in which these numbers are presented. But notice that such an approach cannot explain the *mean rule*, a key rule in practice, which indeed can be obtained, no matter how the data are ordered, by means of a sequential application of binary operators (all of them being different weighted means, each one properly defined depending on the number of aggregated items at each stage).

The model developed in [4], in order to analyze what an *OWA rule* was (see [21,22]), leads to a more general problem about which rules (connective sequences) can be explained by means of binary connectives, to be sequentially applied to arbitrary families of data, properly arranged in advance. Following [5], we then understand as *recursively consistent* those families

$$\{\phi_n : [0, 1]^n \rightarrow [0, 1]\}_{n>1}$$

that can be obtained by means of a binary-based (one by one) recursive procedure. These families of connectives then define a *recursive rule*, conceived as a consistent family of connectives capable of solving arbitrary dimension problems by binary recursion.

The possibility of previous data re-arrangement was kept in the recursive model.

Definition 1.1. If we denote

$$\pi_n(a_1, a_2, \dots, a_n) = (a_{\pi_n(1)}, a_{\pi_n(2)}, \dots, a_{\pi_n(n)}),$$

then an ordering rule π is a consistent family of permutations $\{\pi_n\}_{n>1}$; if for any possible finite collection of numbers when an extra real number a_{n+1} is considered, then

$$\pi_{n+1}(a_1, a_2, \dots, a_n, a_{n+1}) = (a_{\pi_n(1)}, \dots, a_{\pi_n(j-1)}, a_{\pi_{n+1}(j)}, a_{\pi_n(j)}, \dots, a_{\pi_n(n)})$$

holds for some $j \in \{1, \dots, n+1\}$.

Notice that the acceptance of a linear order on the unit interval does define an ordering rule. An ordering rule can just be given by the order in which data are received (real time), which generally cannot be

associated to any particular order in the unit interval. Under this identity ordering rule, the relative position of the previously ordered elements is kept throughout the process.

Now we can formalize the key concept of recursiveness (see [5]).

Definition 1.2. A left-recursive connective rule is a family of connective operators

$$\{\phi_n : [0, 1]^n \rightarrow [0, 1]\}_{n>1}$$

such that there exists a sequence of binary operators

$$\{L_n : [0, 1]^2 \rightarrow [0, 1]\}_{n>1},$$

verifying

$$\phi_2(a_1, a_2) = L_2(a_{\pi(1)}, a_{\pi(2)}), \quad (a_1, \dots, a_n) = L_n(\phi_{n-1}(a_{\pi(1)}, \dots, a_{\pi(n-1)}), a_{\pi(n)}) \quad \forall n > 2$$

for some ordering rule π .

The above definition shows a left-recursive representation, but such a representation is equivalent to the existence of a right-recursive representation, that may be based upon a different ordering rule (see [5]). If both the left and right recursiveness hold for the same ordering rule, then we have a proper *recursive rule* with respect to such an ordering rule. *Commutative* recursive rules will be those recursive rules not dependent on any particular ordering rule, i.e., the aggregation of an arbitrary number of items remains the same no matter what the permutation we apply to the data in advance (some authors refer to this property as *symmetry*; see [18]). *Standard* recursive rules will be those based upon the identity ordering rule (with no data re-arrangement).

Recursiveness appears in principle as an alternative assumption to associativity, weaker but still keeping some of the operational arguments supporting binary operators. Of course some other families of connectives, traditionally considered in the literature as *rules*, will not allow our recursive representation. For example, the *median rule* (see, e.g., [24]): each median can be recursively obtained by means of binary connectives, but not solely by means of *median* binary operators. Some other approaches, allowing alternative characterizations to some of the aggregation rules obtained in this paper, can be found in [10,11,18,19].

Associativity will indeed appear as a particular case of recursiveness, when all the binary operators are the same, both for left and right recursion with the same ordering rule.

Yet, the recursive representation of a connective rule, if any, may be fixed from the underlying ordering rule, as shown in the following result (see [5] for a proof): consistent lower dimension connectives are fixed for any continuous, strictly increasing connective of a recursive rule.

Theorem 1.1. *Let*

$$\{\phi_n : [0, 1]^n \rightarrow [0, 1]\}_{n>1}$$

be a left-recursive connective rule with respect to ordering π with ϕ_n surjective, continuous and strictly increasing in each coordinate, for all n . Then $\{L_n\}_{n>1}$ are unique in their range for such ordering rule π , such that $L_2 = \phi_2$ and

$$\phi_n(a_1, \dots, a_n) = L_n(\phi_{n-1}(a_{\pi(1)}, \dots, a_{\pi(n-1)}), a_{\pi(n)}) \quad \forall n > 2.$$

(The analogous result holds for right recursiveness.)

This paper shows some representative results for recursive rules by noticing that a connective rule $\{\phi_n\}_{n>1}$ is recursive if and only if, a set of general associativity equations (in the sense of Mak [17]) hold for each n , once the items have been properly ordered.

In fact recursiveness holds whenever

$$\phi_n(a_1, \dots, a_n) = R_n(a_{\pi(1)}, \phi_{n-1}(a_{\pi(2)}, \dots, a_{\pi(n)})) = L_n(\phi_{n-1}(a_{\pi(1)}, \dots, a_{\pi(n-1)}), a_{\pi(n)})$$

for all n and some ordering rule π . If each one of these binary connectives L_n, R_n can be assumed to be defined in the cartesian product of two non-trivial compact intervals on the real line, all of them being continuous and strictly increasing in each coordinate, then it can be shown (see [16]) that they are commutative and basically additive in such a way that it takes the expression

$$\phi_n(a_1, \dots, a_n) = \psi^{-1}(\psi(a_1) + \dots + \psi(a_n)).$$

Koopmans's result can allow a particular representation of Theorem 1.1 for additive solutions *in range* (see also [20]). If we take, for example, the natural decreasing order as the underlying ordering rule, then each L_j is defined on a simplex $a_{j-1} \geq a_j$, so that the properties of L_j should theoretically be restricted to this range. Assuming that the above conditions in a properly extended cartesian product of two non-trivial compact intervals, plus continuity and strict increasingness, would assure such a basically additive solution, as it will be shown.

Before presenting our results, it may be beneficial to review some key concepts in the following section.

2. Mak's key assumptions

Let us review some of the key definitions given in Mak [17] in order to have a better understanding of the subsequent results.

Definition 2.1. A generalized groupoid is a quadruple $(X, Y, Z; f)$ of three non-empty sets and a surjective two variable function $f : X \times Y \rightarrow Z$. Regarding the function f as a groupoid product, we will identify a generalized groupoid $(X, Y, Z; f)$ simply by f .

Definition 2.2. (J, f) is a quasigroup if for any given elements a, b, c, d of J equations of the form $f(x, b) = c$ and $f(a, y) = d$ have unique solutions x, y respectively in J .

Definition 2.3. A generalized groupoid $(X, Y, Z; f)$ is called a continuous generalized groupoid (CGG in short) if the three sets involved are non-trivial compact intervals of the real line and f is continuous.

In particular in order to assure, we can apply Mak's representation result [17], we shall restrict ourselves to a certain family of recursive connective rules: the family of connective rules such that Mak's assumptions do apply.

Definition 2.4. A regular recursive connective rule is a family of connective operators

$$\{\phi_n : [0, 1]^n \rightarrow [0, 1]\}_{n>1}$$

such that there exists a sequence of binary operators

$$\{L_n : [0, 1]^2 \rightarrow [0, 1]\}_{n>1}$$

and

$$\{R_n : [0, 1]^2 \rightarrow [0, 1]\}_{n>1},$$

which are continuous generalized groupoids verifying the following conditions:

1. If $x' \leq x''$ and $y' \leq y''$, then $L_n(x', y') \leq L_n(x'', y'')$ and $R_n(x', y') \leq R_n(x'', y'')$.
 If $x' < x''$ and $y' < y''$, then $L_n(x', y') < L_n(x'', y'')$ and $R_n(x', y') < R_n(x'', y'')$.
2. If $x' < x''$, then $L_n(x', y) < L_n(x'', y)$ for some y and $R_n(x', y) < R_n(x'', y)$ for some y .
 If $y' < y''$, then $L_n(x, y') < L_n(x, y'')$ for some x and $R_n(x, y') < R_n(x, y'')$ for some x .
3. $\bar{x}x_i, x'_i \in (0, 1)/L_n(x_i, \bar{x}) = R_n(x'_i, \bar{x}) = 0 \quad \forall \bar{x} \in (0, 1)$.
 $\bar{x}x_i, x'_i \in (0, 1)/L_n(\bar{x}, x_i) = R_n(\bar{x}, x'_i) = 0 \quad \forall \bar{x} \in (0, 1)$.
4. $L_n(0, y') = 0 \quad \forall y' \in [0, 1] \iff L_n(y', 0) = 0 \quad \forall y' \in [0, 1]$.
 $R_n(0, y'') = 0 \quad \forall y'' \in [0, 1] \iff R_n(y'', 0) = 0 \quad \forall y'' \in [0, 1]$.
 $L_n(1, y') = 1 \quad \forall y' \in [0, 1] \iff L_n(y', 1) = 1 \quad \forall y' \in [0, 1]$.
 $R_n(1, y'') = 1 \quad \forall y'' \in [0, 1] \iff R_n(y'', 1) = 1 \quad \forall y'' \in [0, 1]$.

3. Additive rules

First of all, we shall characterize the family of rules founded upon basically additive aggregation operators, as obtained from Mak’s theorem [17]. First, we consider standard rules, so that the ordering rule does not play any role. Generalization when an ordering rule is present will be direct.

Notice that $\{\phi_n : [0, 1]^n \rightarrow [0, 1]\}_{n>1}$ being a regular standard recursive rule, if ϕ_n is strictly increasing for all $n > 1$, then L_n and R_n are strictly increasing binary operators for all $n > 1$.

Theorem 3.1. *Let*

$$\{\phi_n : [0, 1]^n \rightarrow [0, 1]\}_{n>1}$$

be a regular standard recursive rule. If ϕ_n is strictly increasing in each coordinate for all $n > 1$, then there exist:

- (a) $p : [0, 1] \rightarrow \mathbb{R}_+$, *continuous and strictly increasing function,*
 - (b) $\{\delta_n : [0, 1] \rightarrow \mathbb{R}_+\}_{n>1}$, *family of continuous and strictly increasing functions, and*
 - (c) $\{c_n\}_{n \geq 1}$, *sequence of positive real numbers*
- in such a way that*

$$\phi_n(a_1, \dots, a_n) = \delta_n^{-1} \left(\prod_{j=2}^{n-2} c_j \cdot \sum_{k=1}^n c_1^{k-1} p(a_k) \right) \quad \forall (a_1, \dots, a_n) \in [0, 1]^n, \quad \forall n \geq 2, \tag{1}$$

where $\prod_{j=2}^{\ell} c_j$ is taken as 1 whenever $\ell \leq 2$.

Proof. Obviously, from the definition $\{\phi_n\}_{n>1}$ is a standard recursive rule, if and only if, for all $n \geq 3$, the following generalized associativity equation (in the sense of Mak) holds:

$$L_n(R_{n-1}(u, v), w) = R_n(u, L_{n-1}(v, w)). \tag{2}$$

Therefore, having $(a_1, \dots, a_n) \in [0, 1]^n$, taking $u = a_1$, $v = \phi_{n-2}(a_2, \dots, a_{n-2})$ and $w = a_n$ assures the above equation.

Keeping in mind the above relation, we know from [17] that under Mak's assumptions (i.e., $\{L_n\}_{n \geq 1}$ and $\{R_n\}_{n \geq 1}$ binary operators, continuous and strictly increasing in proper compact non-empty intervals) the solution for Eq. (2) is basically additive (see also [16]). That is, there exist $\sigma_n, \theta_n, \delta_n, p_n, q_n$ and r_n , continuous and strictly increasing functions over the compact interval $[0, 1]$, which verify

$$R_{n-1}(u, v) = \sigma_n^{-1}(p_n(u) + q_n(v)), \quad (3)$$

$$L_{n-1}(v, w) = \theta_n^{-1}(q_n(v) + r_n(w)), \quad (4)$$

$$R_n(u, b) = \delta_n^{-1}(p_n(u) + \theta_n(b)), \quad (5)$$

$$L_n(a, w) = \delta_n^{-1}(\sigma_n(a) + r_n(w)). \quad (6)$$

Next, we will prove using induction on n that there exist $\{c_n\}_{n \geq 1}$, $\{\delta_n\}_{n \geq 2}$ and p in the theorem conditions such that for all $n \geq 3$,

$$\theta_n(x) = c_1 c_{n-2} \delta_{n-1}(x),$$

$$\sigma_n(x) = c_{n-2} \delta_{n-1}(x),$$

$$p_n(x) = \prod_{j=2}^{n-2} c_j \cdot p(x),$$

$$q_n(x) = c_1 c_{n-2} \sigma_{n-1}(x),$$

$$r_n(x) = c_1^{n-1} \prod_{j=2}^{n-2} c_j \cdot p(x).$$

In particular, taking $p \equiv c_1 \sigma_2$, we shall prove that for $n = 3$

$$\theta_3(x) = c_1^2 \delta_2(x),$$

$$\sigma_3(x) = c_1 \delta_2(x),$$

$$p_3(x) = p(x),$$

$$q_3(x) = c_1 p(x),$$

$$r_3(x) = c_1^2 p(x).$$

Let us consider

$$L_3(R_2(u, v), w) = R_3(u, L_2(v, w)),$$

where $R_2(a_1, a_2) = \phi_2(a_1, a_2) = L_2(a_1, a_2)$, for all (a_1, a_2) . That way, from (3) and (4) it is concluded that $\{\sigma_3, p_3, q_3\}$ and $\{\theta_3, q_3, r_3\}$ are solutions for the functional equation

$$\phi_2(x, y) = f^{-1}(g(x) + h(y)).$$

Therefore, there exist a, b, c , real numbers such that the following relation is verified:

$$\theta_3(x) = c \sigma_3(x) + a + b,$$

$$q_3(x) = c p_3(x) + a,$$

$$r_3(x) = c q_3(x) + b.$$

Without the loss of generality, $\sigma_3(0) = p_3(0) = q_3(0) = 0$ and $\theta_3(0) = q_3(0) = r_3(0) = 0$ can be assumed. Otherwise, if the three first ones are not equal to 0, then it is possible to define

$$\tilde{\sigma}_3(x) = \sigma_3(x) - \sigma_3(0), \quad \tilde{p}_3(x) = p_3(x) - p_3(0) \quad \text{and} \quad \tilde{q}_3(x) = q_3(x) - q_3(0).$$

Let us check that

$$\phi_2(u, v) = R_2(u, v) = \tilde{\sigma}_3^{-1}(\tilde{p}_3(u) + \tilde{q}_3(v))$$

still holds. Notice that

$$\tilde{\sigma}_3(R_2(u, v)) = \sigma_3(R_2(u, v)) - \sigma_3(0).$$

On the other hand,

$$\tilde{p}_3(u) + \tilde{q}_3(v) = p_3(u) + q_3(v) - p_3(0) - q_3(0).$$

If $\phi_2(0, 0) = 0$, then

$$\sigma_3(0) = \sigma_3(R_2(0, 0)) = p_3(0) + q_3(0).$$

Therefore,

$$\tilde{\sigma}_3^{-1}(\tilde{p}_3(u) + \tilde{q}_3(v)) = R_2(u, v).$$

Consequently, $a = b = 0$. Therefore, $\theta_3(x) = c\sigma_3(x)$, $q_3(x) = cp_3(x)$ and $r_3(x) = cq_3(x)$. Assuming $p \equiv p_3$, $\sigma_3 \equiv c\delta_2$ and $c_1 = c$, then it is true for $n = 3$. Particularly,

$$L_3(a, w) = \delta_3^{-1}(\delta_2(a) + c_1^2 p(w)), \tag{7}$$

$$R_3(u, b) = \delta_3^{-1}(p(u) + c_1 \delta_2(b)). \tag{8}$$

To finish, let us check that if the recurrent expression holds for $n - 1$, then likewise for n . From the general solution of Mak's equation for $n - 1$, it follows that

$$R_{n-1}(u, b) = \delta_{n-1}^{-1}(p_{n-1}(u) + \theta_{n-1}(b)), \tag{9}$$

$$L_{n-1}(a, w) = \delta_{n-1}^{-1}(\sigma_{n-1}(a) + r_{n-1}(w)). \tag{10}$$

And for n , the expressions result to

$$R_{n-1}(u, v) = \sigma_n^{-1}(p_n(u) + q_n(v)),$$

$$L_{n-1}(v, w) = \theta_n^{-1}(q_n(v) + r_n(w)),$$

where following the scheme used above, and considering that

$$L_{n-1}(0, 0) = L_{n-1}(\phi_{n-2}(0, \dots, 0), 0) = 0$$

and

$$R_{n-1}(0, 0) = R_{n-1}(0, \phi_{n-2}(0, \dots, 0)) = 0,$$

there exist $k_1, k_2 \in \mathbb{R}$ such that

$$\begin{aligned}\sigma_n(x) &= k_1 \delta_{n-1}(x), \\ p_n(x) &= k_1 p_{n-1}(x), \\ q_n(x) &= k_1 \theta_{n-1}(x),\end{aligned}$$

whereas

$$\begin{aligned}\theta_n(x) &= k_2 \delta_{n-1}(x), \\ q_n(x) &= k_2 \sigma_{n-1}(x), \\ r_n(x) &= k_2 r_{n-1}(x).\end{aligned}$$

By induction hypothesis, it follows

$$\theta_{n-1}(x) = c_1 c_{n-3} \delta_{n-2}(x) = c_1 \sigma_{n-1}(x).$$

Therefore, from the expressions of q_n , it follows that $k_2 = c_1 k_1$. Hence, assuming $c_{n-2} = k_1$ and using induction hypothesis to deduce p_{n-1} and r_{n-1} from p and $\{c_1, \dots, c_{n-3}\}$, the partial result that was to be proved is satisfied.

The condition imposed about the positiveness of the constants c_n , $n \geq 1$, is justified by the fact that

$$\theta_n(0) = \sigma_n(0) = p_n(0) = q_n(0) = r_n(0) = 0 \quad \forall n \geq 3$$

those being strictly increasing functions, which assure the non-negativeness of all the functions we have been dealing with.

Once we have obtained the expression for the binary operators $\{R_n\}_{n>1}$ and $\{L_n\}_{n>1}$ depending on

$$\{c_n\}_{n \geq 1}, \quad \{\delta_n\}_{n \geq 2}, \quad p,$$

expression (1) of ϕ_n is trivially inferred using induction on n by applying the recursive definition of ϕ_n , thus proving the theorem. \square

Theorem 3.2 (Representation uniqueness). *Let $\{\phi_n : [0, 1]^n \rightarrow [0, 1]\}_{n>1}$, be a recursive connective rule, in the above theorem conditions. $\{L_n\}_{n>1}$ and $\{R_n\}_{n>1}$, are unique in their range.*

Proof. Analogous to Theorem 1.1. \square

The result established in the above Theorem 3.1 can be extended to the case in which the recursive rule is not necessarily standard: if the rule is not standard, then let π be the ordering rule such that

$$\phi_n(a_1, \dots, a_n) = R_n(a_{\pi(1)}, L_{n-1}(\phi_{n-2}(a_{\pi(2)}, \dots, a_{\pi(n-1)}), a_{\pi(n)})) \quad (11)$$

$$= L_n(R_{n-1}(a_{\pi(1)}, \phi_{n-2}(a_{\pi(2)}, \dots, a_{\pi(n-1)}), a_{\pi(n)})) \quad (12)$$

hold $\forall n \geq 3$.

We can consider

$$F_n : S_n \rightarrow [0, 1]$$

with

$$S_n = \{(\pi(x_1), \dots, \pi(x_n)) \mid (x_1, \dots, x_n) \in [0, 1]^n\}$$

in such a way that

$$F_n(\pi(x_1), \dots, \pi(x_n)) = \phi_n(x_1, \dots, x_n) \quad \forall (\pi(x_1), \dots, \pi(x_n)) \in S_n, \quad \forall n > 1.$$

Let F_n be extended over $[0, 1]^n$. For example, if ϕ_n is an OWA operator

$$\phi_n(x_1, \dots, x_n) = w_{1,n}x_{[1]} + \dots + w_{n,n}x_{[n]} \quad \forall (x_1, \dots, x_n) \in [0, 1]^n,$$

then it would be

$$F_n(x_1, \dots, x_n) = w_{1,n}x_1 + \dots + w_{n,n}x_n \quad \forall (x_1, \dots, x_n) \in [0, 1]^n.$$

Assuming that $\{F_n\}_{n \geq 2}$, $\{L_n\}_{n \geq 2}$, $\{R_n\}_{n \geq 2}$ (L_n , R_n extended in the same way) verify the recursiveness equation (11) we have

$$\begin{aligned} F_n(x_1, \dots, x_n) &= R_n(x_1, L_{n-1}(F_{n-2}(x_2, \dots, x_{n-1}), x_n)) \\ &= L_n(R_{n-1}(x_1, F_{n-2}(x_2, \dots, x_{n-1}), x_n)) \quad \forall (x_1, \dots, x_n) \in [0, 1]^n. \end{aligned}$$

The results in the subsequent sections are pursued in order to obtain axiomatic characterization of a few particular aggregation rules that are relevant in the literature, always within the above basically additive framework. It will be shown that by introducing additional properties to the recursive connective rule, the free parameters $\{\delta_n\}_{n > 1}$, $\{c_n\}_{n \geq 1}$ and p defining the rule it would be perfectly set up.

4. Idempotent additive rules

The first property we are going to add is *idempotency*, which is a common assumption in many general contexts. For example, in group decision making, where it assures a highly desired behaviour when all group members agree in the same value, such a value becomes the aggregated value.

Theorem 4.1. *Let*

$$\{\phi_n : [0, 1]^n \rightarrow [0, 1]\}_{n > 1}$$

be a regular standard recursive rule such that ϕ_n is strictly increasing and idempotent for all $n > 1$. Then there exists $p : [0, 1] \rightarrow \mathbb{R}_+$, continuous and strictly increasing function, and a real number $c > 0$ such that

$$\phi_n(a_1, \dots, a_n) = \begin{cases} p^{-1}\left(\frac{1-c}{1-c^n} \sum_{k=1}^n c^{k-1} p(a_k)\right) & \forall (a_1, \dots, a_n) \in [0, 1]^n, \quad \forall n \geq 2 \quad \text{if } c \neq 1, \\ p^{-1}\left(\frac{1}{n} \sum_{k=1}^n p(a_k)\right) & \forall (a_1, \dots, a_n) \in [0, 1]^n, \quad \forall n \geq 2 \quad \text{if } c = 1. \end{cases}$$

Proof. From Theorem 3.1

$$\phi_n(a_1, \dots, a_n) = \delta_n^{-1} \left(\prod_{j=2}^{n-2} c_j \cdot \sum_{k=1}^n c_1^{k-1} p(a_k) \right) \quad \forall (a_1, \dots, a_n) \in [0, 1]^n, \quad \forall n \geq 2.$$

Then, since ϕ_n is an idempotent aggregator for all $a \in [0, 1]$,

$$a = \phi_n(a, \dots, a) = \delta_n^{-1}(e_n p(a)),$$

being

$$e_n = \prod_{j=2}^{n-2} c_j \cdot \sum_{k=1}^n c_1^{k-1} = \begin{cases} \frac{1-c_1^n}{1-c_1} \prod_{j=2}^{n-2} c_j & \text{if } c_1 \neq 1, \\ n \prod_{j=2}^{n-2} c_j & \text{if } c_1 = 1. \end{cases}$$

That means, $\delta_n \equiv e_n p$. Then, the family of functions $\{\delta_n\}_{n>1}$ is determined from the sequence of real numbers $\{c_n\}_{n \geq 1}$ and the unique function p . Furthermore,

$$\begin{aligned} \phi_n(a_1, \dots, a_n) &= p^{-1} \left(\frac{1}{e_n} \prod_{j=2}^{n-2} c_j \cdot \sum_{k=1}^n c_1^{k-1} p(a_k) \right) = p^{-1} \left(\frac{1-c_1}{1-c_1^n} \sum_{k=1}^n c_1^{k-1} p(a_k) \right), \quad c_1 \neq 1, \\ \phi_n(a_1, \dots, a_n) &= p^{-1} \left(\frac{1}{n} \sum_{k=1}^n p(a_k) \right), \quad c_1 = 1, \end{aligned}$$

for all $(a_1, \dots, a_n) \in [0, 1]^n$ for all $n \geq 2$. Then, taking $c = c_1$, ϕ_n exclusively depends on c and p .

Taking into account the recurrent expression of the following functions $\theta_n, \sigma_n, p_n, q_n$ and r_n , for $n \geq 2$ that was obtained in the proof of Theorem 3.1, it follows that the binary operators $\{L_n\}_{n>1}$ and $\{R_n\}_{n>1}$ can be expressed as functions of c and p :

$$\begin{aligned} L_n(a, b) &= \delta_n^{-1}(\sigma_n(a) + r_n(b)) = p^{-1} \left(\frac{\sigma_n(a)}{e_n} + \frac{r_n(b)}{e_n} \right) \\ &= p^{-1} \left(\frac{c_{n-2} e_{n-1}}{e_n} p(a) + \frac{c_1^{n-1} \prod_{j=2}^{n-2} c_j}{e_n} p(b) \right) = p^{-1} \left(\frac{1-c^{n-1}}{1-c^n} p(a) + \frac{c^{n-1}(1-c)}{1-c^n} p(b) \right) \end{aligned}$$

and

$$\begin{aligned} R_n(a, b) &= \delta_n^{-1}(p_n(a) + \theta_n(b)) = p^{-1} \left(\frac{p_n(a)}{e_n} + \frac{\theta_n(b)}{e_n} \right) \\ &= p^{-1} \left(\frac{\prod_{j=2}^{n-2} c_j}{e_n} p(a) + \frac{c_1 c_{n-2} e_{n-1}}{e_n} p(b) \right) = p^{-1} \left(\frac{1-c}{1-c^n} p(a) + \frac{c(1-c^{n-1})}{1-c^n} p(b) \right) \end{aligned}$$

if $c_1 \neq 1$, whereas

$$\begin{aligned} L_n(a, b) &= p^{-1} \left(\left(1 - \frac{1}{n}\right) p(a) + \frac{1}{n} p(b) \right), \\ R_n(a, b) &= p^{-1} \left(\frac{1}{n} p(a) + \left(1 - \frac{1}{n}\right) p(b) \right) \end{aligned}$$

if $c_1 = 1$. \square

Notice that $\{\phi_n : [0, 1]^n \rightarrow [0, 1]\}_{n>1}$ being a rule verifying the above theorem conditions, for each $n > 2$, there exists a weight system $w_{i,n}$, $i = 1, \dots, n$, and a function p such that

$$\phi_n(a_1, \dots, a_n) = p^{-1}(w_{1,n} p(a_1) + \dots + w_{n,n} p(a_n)) \quad \forall (a_1, \dots, a_n) \in [0, 1]^n,$$

where

(i) For all $i = 1, \dots, n$, and for all $n \geq 2$,

$$w_{i,n} = \begin{cases} \frac{1-c}{1-c^n} c^{i-1} = \frac{1-c^{n+1}}{1-c^n} w_{i,n+1} \in (0, 1) & \text{if } c \neq 1, \\ \frac{1}{n} = \frac{n+1}{n} w_{i,n+1} \in (0, 1) & \text{if } c = 1. \end{cases}$$

(ii) $\sum_{i=1}^n w_{i,n} = 1$.

Furthermore $\forall n > 1$ there exist

$$\ell_n = \begin{cases} \frac{1-c^{n-1}}{1-c^n} \in (0, 1) & \text{if } c \neq 1, \\ 1 - \frac{1}{n} \in (0, 1) & \text{if } c = 1, \end{cases}$$

and

$$r_n = \begin{cases} \frac{1-c}{1-c^n} \in (0, 1) & \text{if } c \neq 1, \\ \frac{1}{n} \in (0, 1) & \text{if } c = 1, \end{cases}$$

such that

$$\begin{aligned} L_n(a, b) &= p^{-1}(\ell_n p(a) + (1 - \ell_n)p(b)), \\ R_n(a, b) &= p^{-1}(r_n p(a) + (1 - r_n)p(b)). \end{aligned}$$

The above shows that every regular standard recursive rule that is strictly increasing and idempotent is made up of aggregation operators that are *quasi-linear means* with *generator function* p and weights ℓ_n , $1 - \ell_n$ and r_n , $1 - r_n$ for L_n and R_n , respectively (see [6,7,18]).

When dealing with non-standard rules, the above theorem can be extended as follows:

Theorem 4.2. *Let*

$$\{\phi_n : [0, 1]^n \rightarrow [0, 1]\}_{n>1}$$

be a regular recursive rule and π the underlying ordering rule. If ϕ_n is strictly increasing and idempotent for all $n > 1$, then there exists $p : [0, 1] \rightarrow \mathbb{R}_+$, continuous and strictly increasing function, and a real number $c > 0$ such that

$$\phi_n(a_1, \dots, a_n) = p^{-1} \left(\frac{1-c}{1-c^n} \sum_{k=1}^n c^{k-1} p(a_{\pi(k)}) \right) \quad \forall (a_1, \dots, a_n) \in [0, 1]^n, \quad \forall n \geq 2. \tag{13}$$

Notice that the case $c = 1$ has been omitted, since in such a case, $\{\phi_n : [0, 1]^n \rightarrow [0, 1]\}_{n>1}$ is a commutative rule.

Being π the natural decreasing order, a recursive rule family of special interest shows up. Concretely, being $\{\phi_n : [0, 1]^n \rightarrow [0, 1]\}_{n>1}$ the regular recursive rule and π the underlying ordering rule, all the connective operators are *quasi-OWA* operators (see [10]).

Remark 4.1. Notice that when idempotency is imposed over all the recursive rule elements, the way to aggregate whichever n items is set up once the aggregation of two items has been decided. All the free parameters are set up out of ϕ_2 . For example, if ϕ_2 is an OWA operator then the whole rule is settled of OWA operators, and the list of weights is univocally determined.

5. Homogeneous additive rules

Now we will deal with the homogeneous functions of degree one. The homogeneous property intuitively says that a variation of all the degrees of membership by the same factor would change the aggregated degree by this factor. We will see that being ϕ_n a homogeneous function of degree one, that is, if

$$\phi_n(ka_1, \dots, ka_n) = k\phi_n(a_1, \dots, a_n) \quad \forall (a_1, \dots, a_n), \forall k, \forall n,$$

then the recursive rule has two degrees of freedom.

Theorem 5.1. *Let*

$$\{\phi_n : [0, 1]^n \rightarrow [0, 1]\}_{n>1}$$

be a regular standard recursive rule. If ϕ_n is a homogenous strictly increasing function and idempotent, for all $n > 1$, then there exist two real numbers $c > 0$ and $r > 0$ such that

$$\phi_n(a_1, \dots, a_n) = \left(\frac{1-c}{1-c^n} \sum_{k=1}^n c^{k-1} a_k^r \right)^{1/r} \quad \forall (a_1, \dots, a_n) \in [0, 1]^n, \forall n \geq 2. \quad (14)$$

Proof. Theorem 4.1 shows that every regular standard recursive rule with $\{\phi_n\}_{n>1}$, being an strictly increasing and idempotent function, is composed by quasilinear means. Hence,

$$\phi_n(a_1, \dots, a_n) = p^{-1} \left(\frac{1-c}{1-c^n} \sum_{k=1}^n c^{k-1} p(a_k) \right) \quad \forall (a_1, \dots, a_n) \in [0, 1]^n, \forall n \geq 2.$$

Since ϕ_n is homogeneous, there exist $r > 0$ and $c > 0$ such that $p(t) = t^r$ (see [14]). Therefore

$$\phi_n(a_1, \dots, a_n) = \left(\frac{1-c}{1-c^n} \sum_{k=1}^n c^{k-1} a_k^r \right)^{1/r} \quad \forall (a_1, \dots, a_n) \in [0, 1]^n, \forall n \geq 2. \quad \square$$

If $\{\phi_n : [0, 1]^n \rightarrow [0, 1]\}_{n>1}$ is not standard, then the above results can be established as follows:

Theorem 5.2. *Let*

$$\{\phi_n : [0, 1]^n \rightarrow [0, 1]\}_{n>1}$$

be a regular recursive rule and let π be the underlying ordering rule. If ϕ_n is an strictly increasing, idempotent and homogeneous function, then for all $n > 1$ there exist $c > 0$ and $r > 0$ such that

$$\phi_n(a_1, \dots, a_n) = \left(\frac{1-c}{1-c^n} \sum_{k=1}^n c^{k-1} a_{\pi(k)}^r \right)^{1/r} \quad \forall (a_1, \dots, a_n) \in [0, 1]^n, \forall n \geq 2. \quad (15)$$

Remark 5.1. In all the results obtained in this section, the condition that the connective operators are strictly increasing imposed on the rule $\{\phi_n\}_{n>1}$ can be loosened. It is enough just asking ϕ_2 to be strictly increasing: since the additivity of the solution for the generalized associativity equation of order n , depends

exclusively on the binary operators R_{n-1} and L_{n-1} (see [17]). ϕ_2 being a strictly increasing function, implies the additivity of equation of order 3. Hence, ϕ_3, L_3 and R_3 must be strictly increasing. Besides, the solution for the equation of order 4 is also additive. Therefore, using the same argument iteratively, we can conclude that solutions for all the generalized associativity equations are of the additive type and ϕ_n is a strictly increasing function for all $n \geq 3$.

6. Geometric rules

Still taking advantage of the key theorem in Mak [17], now we will show some particular basically additive solutions by considering alternative additional conditions, this time related to boundary behaviour. We shall notice that in this context the natural representation seems to be multiplicative.

The following key definition was given in Mak [17].

Definition 6.1. $f : [0, 1]^2 \rightarrow [0, 1]$ is an ordinal sum of strict t -norms if and only if there exists a countable collection of disjoint open intervals in $[0, 1]$, $\{I_\alpha\}_{\alpha \in A}$, and (J_α, f_α) , semigroup collection, being $J_\alpha = \bar{I}_\alpha = [\underline{x}, \bar{x}]$, and $f_\alpha : J_\alpha^2 \rightarrow J_\alpha$ a function defined by

$$f_\alpha(x, y) = j_\alpha^{-1}(j_\alpha(x) \cdot j_\alpha(y)) \quad \forall (x, y) \in J_\alpha^2,$$

where $j_\alpha : J_\alpha \rightarrow [0, 1]$ is a continuous, strictly increasing function with $j_\alpha(\underline{x}) = 0$ and $j_\alpha(\bar{x}) = 1$ such that

$$f(x, y) = \begin{cases} j_\alpha^{-1}(j_\alpha(x) \cdot j_\alpha(y)) & \text{if } (x, y) \in J_\alpha \text{ for some } \alpha \in A, \\ \min\{x, y\} & \text{otherwise.} \end{cases}$$

The terminology used nowadays for such an “ordinal addition of isotopic to the probabilistic t -norm semigroup” is an “ordinal sum of strict t -norms” (see, e.g., [13,15]).

Theorem 6.1. *Let*

$$\{\phi_n : [0, 1]^n \rightarrow [0, 1]\}_{n \geq 1}$$

be a regular standard recursive rule. If $\phi_2 : [0, 1]^2 \rightarrow [0, 1]$ verifies the following boundary conditions:

- (i) $\phi_2(0, x) = \phi_2(x, 0) = 0 \quad \forall x \in [0, 1]^n$,
- (ii) $\phi_2(x, 1) = \phi_2(1, x) \quad \forall x \in [0, 1]^n$,

then there exist

- (a) $\psi : [0, 1] \rightarrow [0, 1]$ automorphism,
- (b) $\{\delta_n : [0, 1] \rightarrow [0, 1]\}_{n \geq 1}$ automorphism family,
- (c) $f^n : [0, 1]^n \rightarrow [0, 1]$, where f^n is defined from $f^2 = f$ just using associativity, being $f : [0, 1]^2 \rightarrow [0, 1]$ and ordinal sum of strict t -norms,

verifying $\forall n \geq 2$:

- $\phi_n(x_1, \dots, x_n) = \delta_n^{-1}(f^n(\psi(x_1), \dots, \psi(x_n)))$,
- $L_n(a, b) = \delta_n^{-1}(f(\delta_{n-1}(a), \psi(b))) = R_n(b, a)$.

Proof. Since $L_2 \equiv \phi_2 \equiv R_2$ and

$$L_2(0, x) = R_2(0, x) = R_2(x, 0) = L_2(x, 0) = 0 \quad \forall x \in [0, 1].$$

Then, according to Mak [17], the solution of equation

$$L_3(R_2(x_1, x_2), x_3) = R_3(x_1, L_2(x_2, x_3))$$

should be such that

$$\begin{aligned} R_2(u, v) &= \sigma_3^{-1}(f(p_3(u), q_3(v))), \\ L_2(u, v) &= \theta_3^{-1}(f(q_3(u), r_3(v))), \\ L_3(a, b) &= \delta_3^{-1}(f(\sigma_3(a), r_3(b))), \\ R_3(a, b) &= \delta_3^{-1}(f(p_3(a), \theta_3(b))), \end{aligned} \tag{16}$$

being $\sigma_3, \theta_3, \delta_3, p_3, q_3, r_3$ automorphisms and f an ordinal sum of strict t -norms.

First of all, notice that $f(x, 1) = f(1, x) = x \ \forall x \in [0, 1]$. If there exists α_1 such that $1 \in J_{\alpha_1}$, then

$$\begin{aligned} f(x, 1) &= f(1, x) = j_{\alpha_1}^{-1}(j_{\alpha_1}(1) \cdot j_{\alpha_1}(x)) = x \quad \forall x \in J_{\alpha_1}, \\ f(x, 1) &= f(1, x) = \min\{x, 1\} = x \quad \forall x \notin J_{\alpha_1}. \end{aligned}$$

Otherwise, $1 \notin J_{\alpha}$ for all α , and

$$f(1, x) = f(x, 1) = \min\{x, 1\} = x \quad \forall x \in [0, 1].$$

Then

$$\begin{aligned} R_2(x, 1) &= \sigma_3^{-1}(p_3(x)), \\ L_2(x, 1) &= \theta_3^{-1}(q_3(x)), \\ R_2(1, x) &= \sigma_3^{-1}(q_3(x)), \\ L_2(1, x) &= \theta_3^{-1}(r_3(x)), \end{aligned}$$

where, taking into account $\phi_2(x, 1) = \phi_2(1, x)$ for all $x \in [0, 1]$, it follows:

$$\sigma_3^{-1}(p_3(x)) = \theta_3^{-1}(q_3(x)) = \sigma_3^{-1}(q_3(x)) = \theta_3^{-1}(r_3(x)) \quad \forall x \in [0, 1].$$

Since $\sigma_3, \theta_3, \delta_3, p_3, q_3, r_3$ are surjective, then $\sigma_3 \equiv \theta_3$ and $p_3 \equiv q_3 \equiv r_3$. Therefore, taking $\delta_2 \equiv \sigma_3 \equiv \theta_3$ and $\psi \equiv p_3 \equiv q_3 \equiv r_3$, it holds

$$L_2(u, v) = R_2(u, v) = \phi_2(u, v) = \delta_2^{-1}(f(\psi(u), \psi(v))), \quad L_3(a, b) = \delta_3^{-1}(f(\psi(a), \delta_2(b))) = R_3(b, a).$$

Next, it will be tried out by induction on n that the solutions for the generalized associativity equation of order n ,

$$L_n(R_{n-1}(u, v), w) = R_n(u, L_{n-1}(v, w)),$$

are

$$\begin{aligned} L_{n-1}(a, b) &= \delta_{n-1}^{-1}(f(\delta_{n-2}(a), \psi(b))) = R_{n-1}(b, a), \\ L_n(a, b) &= \delta_n^{-1}(f(\delta_{n-1}(a), \psi(b))) = R_n(b, a), \end{aligned}$$

being δ_n and δ_{n-1} automorphisms $\forall n \geq 3$.

It has been proven for $n = 3$. By induction hypothesis

$$R_{n-1}(0, x) = R_{n-1}(x, 0) = 0 \quad \forall x \in [0, 1]$$

and

$$L_{n-1}(0, x) = L_{n-1}(x, 0) = 0 \quad \forall x \in [0, 1].$$

Then the solution for the n th generalized associativity equation would be

$$\begin{aligned} R_{n-1}(u, v) &= \sigma_n^{-1}(\tilde{f}(p_n(u), q_n(v))), \\ L_{n-1}(v, w) &= \theta_n^{-1}(\tilde{f}(q_n(v), r_n(w))), \\ L_n(a, b) &= \delta_n^{-1}(\tilde{f}(\sigma_n(a), r_n(b))), \\ R_n(a, b) &= \delta_n^{-1}(\tilde{f}(p_n(a), \theta_n(b))), \end{aligned}$$

where according to Theorem 5.11, in Mak [17], $\sigma_n : [0, 1] \rightarrow [0, 1]$ and $\theta_n : [0, 1] \rightarrow [0, 1]$ are any couple of automorphisms such that

$$\sigma_n(R_{n-1}(1, v)) = \theta_n(L_{n-1}(v, 1)) \quad \forall v \in [0, 1], \tag{17}$$

whereas

$$\begin{aligned} p_n(u) &= \sigma_n(R_{n-1}(u, 1)), \\ q_n(v) &= \sigma_n(R_{n-1}(1, v)), \\ r_n(w) &= \theta_n(L_{n-1}(1, w)). \end{aligned}$$

$\delta_n : [0, 1] \rightarrow [0, 1]$ is an automorphism such that $\delta_n(L_n(a, 1)) = \sigma_n(a)$ and

$$\tilde{f}(x, y) = \sigma_n(R_{n-1}(p_n^{-1}(x), q_n^{-1}(y))).$$

Now, it will be tried out that $\sigma_n \equiv \theta_n \equiv \delta_{n-1}$ verify (17). From induction hypothesis it follows:

$$L_{n-1}(u, v) = \delta_{n-1}^{-1}(f(\delta_{n-2}(u), \psi(v))) = R_{n-1}(v, u) \quad \forall (u, v) \in [0, 1]^2,$$

where $\delta_{n-1} : [0, 1] \rightarrow [0, 1]$ and $\delta_{n-2} : [0, 1] \rightarrow [0, 1]$ are two automorphisms. Then,

$$\sigma_n(R_{n-1}(1, v)) = \delta_{n-1}(R_{n-1}(1, v)) = \delta_{n-1}(L_{n-1}(v, 1)) = \theta_n(L_n(v, 1)) \quad \forall v \in [0, 1].$$

Thus

$$\begin{aligned} p_n(u) &= \delta_{n-1}(R_{n-1}(u, 1)) = \delta_{n-1}(\delta_{n-1}^{-1}(f(\psi(u), \delta_{n-2}(1)))) = \psi(u) \quad \forall u \in [0, 1], \\ q_n(v) &= \delta_{n-1}(R_{n-1}(1, v)) = \delta_{n-1}(\delta_{n-1}^{-1}(f(\psi(1), \delta_{n-2}(v)))) = \delta_{n-2}(v) \quad \forall v \in [0, 1], \\ r_n(w) &= \delta_{n-1}(L_{n-1}(1, w)) = \delta_{n-1}(\delta_{n-1}^{-1}(f(\delta_{n-2}(1), \psi(w)))) = \psi(w) \quad \forall w \in [0, 1]. \end{aligned}$$

Therefore for all $(x, y) \in [0, 1]^2$ it holds

$$\begin{aligned} \tilde{f}(x, y) &= \delta_{n-1}(R_{n-1}(p_n^{-1}(x), q_n^{-1}(y))) \\ &= \delta_{n-1}(\delta_{n-1}^{-1}(f(\psi(p_n^{-1}(x)), \delta_{n-2}(q_n^{-1}(y))))) = f(x, y). \end{aligned}$$

Thus, it has been proven that

$$\begin{aligned} R_{n-1}(u, v) &= \delta_n^{-1}(f(\psi(u), \delta_{n-2}(v))) = L_{n-1}(v, u), \\ R_n(a, b) &= \delta_n^{-1}(f(\psi(a), \delta_{n-1}(b))) = L_n(b, a), \end{aligned}$$

where

$$\delta_n : [0, 1] \rightarrow [0, 1]$$

is given by

$$\delta_n(z) = \delta_{n-1}(L_n(z, 1)) \quad \forall z \in [0, 1]. \quad \square$$

Again, when the aggregation operators are strictly increasing, we obtain easier solutions; this time *geometric rules*.

Theorem 6.2. *Let*

$$\{\phi_n : [0, 1]^n \rightarrow [0, 1]\}_{n \geq 1}$$

be a regular standard recursive rule. If $\phi_2 : [0, 1]^2 \rightarrow [0, 1]$ is strictly increasing in $(0, 1]^2$ and $\phi_2(0, x) = \phi_2(x, 0) = 0$, $\forall x \in [0, 1]^n$, then there exist

- (a) $p : [0, 1] \rightarrow [0, 1]$ *an automorphism,*
 - (b) $\{\delta_n : [0, 1] \rightarrow [0, 1]\}_{n \geq 1}$ *automorphism family,*
 - (c) $\{c_n\}_{n \geq 1}$ *a sequence of positive real numbers,*
- verifying*

$$\phi_n(a_1, \dots, a_n) = \delta_n^{-1} \left(\left(\prod_{k=1}^n p(a_k)^{c_1^{k-1}} \right)^{\prod_{j=2}^{n-2} c_j} \right) \quad \forall (a_1, \dots, a_n) \in [0, 1]^n, \quad \forall n \geq 2,$$

where $\prod_{j=2}^{\ell} c_j$ with $\ell \leq 2$ is taken equal to 1.

Proof. First, we will prove that the solution for the generalized associativity equation of order n is basically multiplicative. That means, there exist $\sigma_n, \theta_n, \delta_n, p_n, q_n$ and r_n , automorphisms such that

$$R_{n-1}(u, v) = \sigma_n^{-1}(p_n(u) \cdot q_n(v)), \tag{18}$$

$$L_{n-1}(v, w) = \theta_n^{-1}(q_n(v) \cdot r_n(w)), \tag{19}$$

$$R_n(u, b) = \delta_n^{-1}(p_n(u) \cdot \theta_n(b)), \tag{20}$$

$$L_n(a, w) = \delta_n^{-1}(\sigma_n(a) \cdot r_n(w)). \tag{21}$$

In the theorem conditions, according to Mak [17], the solution for equation

$$L_3(R_2(x_1, x_2), x_3) = R_3(x_1, L_2(x_2, x_3))$$

is

$$\begin{aligned} R_2(u, v) &= \tilde{\sigma}_3^{-1} \left(f(\tilde{p}_3(u), \tilde{q}_3(v)) \right), \\ L_2(u, v) &= \tilde{\theta}_3^{-1} \left(f(\tilde{q}_3(u), \tilde{r}_3(v)) \right), \\ L_3(a, b) &= \tilde{\delta}_3^{-1} \left(f(\tilde{\sigma}_3(a), \tilde{r}_3(b)) \right), \\ R_3(a, b) &= \tilde{\delta}_3^{-1} \left(f(\tilde{p}_3(a), \tilde{\theta}_3(b)) \right), \end{aligned}$$

being $\tilde{\sigma}_3, \tilde{\theta}_3, \tilde{\delta}_3, \tilde{p}_3, \tilde{q}_3, \tilde{r}_3$ automorphism and f an ordinal sum of strict t -norms.

We will show that there exists an automorphism $j : [0, 1] \rightarrow [0, 1]$ such that

$$f(x, y) = j^{-1}(j(x) \cdot j(y)) \quad \forall (x, y) \in [0, 1]^2.$$

If f would not be that way, then $\{I_\alpha\}_{\alpha \in A} \neq \{(0, 1)\}$. Therefore, there would exist $I_\alpha = (\underline{\alpha}, \bar{\alpha})$, with $\underline{\alpha} > 0$ such that

$$f(\underline{\alpha}, x) = f_\alpha(\underline{\alpha}, x) = j_\alpha^{-1}(j_\alpha(\underline{\alpha}) \cdot j_\alpha(x)) = j_\alpha^{-1}(0) = \underline{\alpha} \quad \forall x \in [\underline{\alpha}, \bar{\alpha}]$$

and

$$f(\underline{\alpha}, x) = \min\{\underline{\alpha}, x\} = \underline{\alpha} \quad \forall x \in (\bar{\alpha}, 1].$$

Hence, f would not be strictly increasing in $(0, 1]^2$, neither would $\phi_2 = L_2 = R_2$.

Then, taking

$$\begin{aligned} \sigma_3 &= j \circ \tilde{\sigma}_3, \\ \theta_3 &= j \circ \tilde{\theta}_3, \\ p_3 &= j \circ \tilde{p}_3, \\ q_3 &= j \circ \tilde{q}_3, \\ r_3 &= j \circ \tilde{r}_3, \\ \delta_3 &= j \circ \tilde{\delta}_3, \end{aligned} \tag{22}$$

the assertion holds for $n = 3$. Particularly, L_3 and R_3 are strictly increasing functions in $(0, 1]^2$.

Now, let us prove that the assertion is true for $n - 1$, as it is for n . It follows from induction hypothesis that

$$\begin{aligned} L_{n-1}(u, v) &= \delta_{n-1}^{-1}(\sigma_{n-1}(u) \cdot r_{n-1}(v)), \\ R_{n-1}(u, v) &= \delta_{n-1}^{-1}(p_{n-1}(u) \cdot \theta_{n-1}(v)). \end{aligned}$$

Then, $L_{n-1}(0, v) = L_{n-1}(v, 0) = 0 \quad \forall v \in [0, 1]$ and $R_{n-1}(0, v) = R_{n-1}(v, 0) = 0 \quad \forall v \in [0, 1]$. Thus, according to Mak [17], the solution for the generalized associativity equation of order n is

$$R_{n-1}(u, v) = \tilde{\sigma}_n^{-1} \left(f_n(\tilde{p}_n(u), \tilde{q}_n(v)) \right), \tag{23}$$

$$L_{n-1}(u, v) = \tilde{\theta}_n^{-1} \left(f_n(\tilde{q}_n(u), \tilde{r}_n(v)) \right), \tag{24}$$

$$L_n(a, b) = \tilde{\delta}_n^{-1} \left(f_n(\tilde{\sigma}_n(a), \tilde{r}_n(b)) \right), \tag{25}$$

$$R_n(a, b) = \tilde{\delta}_n^{-1} \left(f_n(\tilde{p}_n(a), \tilde{\theta}_n(b)) \right), \tag{26}$$

being $\tilde{\sigma}_n, \tilde{\theta}_n, \tilde{\delta}_n, \tilde{p}_n, \tilde{q}_n, \tilde{r}_n$ automorphism and f_n an ordinal sum of strict t -norms.

Following the above reasoning, the existence of an automorphism j_n is determined such that

$$f_n(x, y) = j_n^{-1}(j_n(x) \cdot j_n(y)) \quad \forall (x, y) \in [0, 1]^2.$$

Hence, taking

$$\begin{aligned} \sigma_n &= j_n \circ \tilde{\sigma}_n, \\ \theta_n &= j_n \circ \tilde{\theta}_n, \\ p_n &= j_n \circ \tilde{p}_n, \\ q_n &= j_n \circ \tilde{q}_n, \\ r_n &= j_n \circ \tilde{r}_n, \\ \delta_n &= j_n \circ \tilde{\delta}_n, \end{aligned} \tag{27}$$

the assertion holds for n .

Next we will prove by induction on n , that there exist $\{c_n\}_{n \geq 1}$, $\{\delta_n\}_{n \geq 2}$, and p , in the theorem conditions, that

$$\begin{aligned} \theta_n(x) &= \delta_{n-1}^{c_1 \cdot c_{n-2}}(x), \\ \sigma_n(x) &= \delta_{n-1}^{c_{n-2}}(x), \\ p_n(x) &= p(x) \prod_{j=2}^{n-2} c_j, \\ q_n(x) &= \sigma_{n-1}^{c_1 \cdot c_{n-2}}(x), \\ r_n(x) &= p(x) c_1^{n-1} \prod_{j=2}^{n-2} c_j, \end{aligned}$$

for all $n \geq 4$, being for $n = 3$,

$$\begin{aligned} \theta_3(x) &= \delta_2(x)^{c_1}, \\ \sigma_3(x) &= \delta_2(x), \\ p_3(x) &= p(x), \\ q_3(x) &= p(x)^{c_1}, \\ r_3(x) &= p(x)^{c_1^2}. \end{aligned}$$

First, we will show that it holds for $n = 3$ and $n = 4$. Consider the equation

$$L_3(R_2(u, v), w) = R_3(u, L_2(v, w)),$$

where $R_2(u, v) = \phi_2(u, v) = L_2(u, v)$ for all $(u, v) \in [0, 1]^2$. That way, from (18) and (19). Then $\{\sigma_3, p_3, q_3\}$ and $\{\theta_3, q_3, r_3\}$ are solutions for the functional equation

$$\phi_2(x, y) = f^{-1}(g(x) \cdot h(y)) \quad \forall (x, y) \in [0, 1]^2. \tag{28}$$

Therefore

$$\sigma_3^{-1}(p_3(x) \cdot q_3(y)) = \theta_3^{-1}(q_3(x) \cdot r_3(y)).$$

Then

$$\theta_3(\sigma_3^{-1}(p_3(x) \cdot q_3(y))) = q_3(x) \cdot r_3(y). \tag{29}$$

Taking

$$\begin{aligned} p_3(x) &= u, \\ q_3(y) &= v, \\ \theta_3 \circ \sigma_3^{-1}(w) &= \varphi(w), \\ q_3(p_3^{-1}(u)) &= \psi(u) \end{aligned}$$

and

$$r_3(q_3^{-1}(v)) = \chi(v),$$

Eq. (29) can be written as follows:

$$\varphi(u \cdot v) = \psi(u) \cdot \chi(v).$$

Solutions for such an equation are (see [1])

$$\begin{aligned} \varphi(t) &= abt^c, \\ \psi(t) &= at^c, \\ \chi(t) &= bt^c, \end{aligned}$$

or the trivial solution. Since

$$\varphi(1) = \theta_3 \circ \sigma_3^{-1}(1) = 1,$$

we have

$$\begin{aligned} \theta_3(\sigma_3^{-1}(t)) &= abt^c, \\ q_3(p_3^{-1}(t)) &= at^c, \\ r_3(q_3^{-1}(t)) &= bt^c. \end{aligned}$$

Therefore

$$\begin{aligned} \sigma_3^{-1}(t) &= \theta_3^{-1}(abt^c), \\ p_3^{-1}(t) &= q_3^{-1}(at^c), \\ q_3^{-1}(t) &= r_3^{-1}(bt^c). \end{aligned}$$

Since $\sigma_3, \theta_3, \delta_3, p_3, q_3, r_3$ are automorphisms of the unit interval, it follows that $a = 1$ and $b = 1$. Therefore, there exists $c > 0$ such that

$$\begin{aligned} \sigma_3^{-1}(t) &= \theta_3^{-1}(t^c), \\ p_3^{-1}(t) &= q_3^{-1}(t^c), \\ q_3^{-1}(t) &= r_3^{-1}(t^c). \end{aligned}$$

Thus

$$\begin{aligned}\theta_3(z) &= \sigma_3(z)^c, \\ q_3(z) &= p_3(z)^c, \\ r_3(z) &= p_3(z)^{c^2}.\end{aligned}$$

Taking $p \equiv p_3$ and $\delta_2 \equiv \sigma_3$ and $c_1 = c$ the result holds for $n = 3$. Particularly,

$$L_3(a, w) = \delta_3^{-1}(\delta_2(a) \cdot p(w)^{c_1^2}), \quad (30)$$

$$R_3(u, b) = \delta_3^{-1}(p(u) \cdot \delta_2(b)^{c_1}). \quad (31)$$

Now we will show that it is also true for $n = 4$. From the above expression of R_3 (31), and taking into account the general solution of Mak's equation for $n = 4$, it is concluded that $\{\sigma_4, p_4, q_4\}$ and $\{\delta_3, p, \delta_2^{c_1}\}$ are solutions for the functional equation

$$R_3(x, y) = f^{-1}(g(x) \cdot h(y)).$$

Following the same reasoning as before, it follows that there exists $k_1 \in \mathbb{R}$ such that

$$\sigma_4(z) = \delta_3(z)^{k_1}, \quad (32)$$

$$p_4(z) = p(z)^{k_1}, \quad (33)$$

$$q_4(z) = \delta_2(z)^{c_1 \cdot k_1}. \quad (34)$$

Analogously, $\{\theta_4, q_4, r_4\}$ and $\{\delta_3, \delta_2, p^{c_1}\}$ are solutions for

$$L_3(x, y) = f^{-1}(g(x) \cdot h(y))$$

and it follows that there exists $k_2 \in \mathbb{R}$ such that

$$\theta_4(z) = \delta_3(z)^{k_2}, \quad (35)$$

$$q_4(z) = \delta_2(z)^{k_2}, \quad (36)$$

$$r_4(z) = p(z)^{c_1^2 \cdot k_2}. \quad (37)$$

From expressions (34) and (36) it can be deduced that $c \cdot k_1 = k_2$. Therefore taking $c_2 = k_1$, and following the same scheme that we did for $n = 3$, the recurrent expression for $n = 4$ is concluded.

To finish, we verify that if the recurrent expression for $n - 1$ holds, as it does for n . From the general solution of Mak's equation for $n - 1$, it follows:

$$R_{n-1}(u, b) = \delta_{n-1}^{-1}(p_{n-1}(u) \cdot \theta_{n-1}(b)),$$

$$L_{n-1}(a, w) = \delta_{n-1}^{-1}(\sigma_{n-1}(a) \cdot r_{n-1}(w)).$$

On the other hand, it follows from the expression of the Mak equation of order n , that

$$R_{n-1}(u, v) = \sigma_n^{-1}(p_n(u) \cdot q_n(v)),$$

$$L_{n-1}(u, v) = \theta_n^{-1}(q_n(u) \cdot r_n(v)).$$

Following the scheme above, it can be proved that there exist $\delta_{n-1}, p_{n-1}, \theta_{n-1}, \sigma_n, p_n, q_n, k_1$ and k_2 such that

$$\sigma_n(z) = \delta_{n-1}(z)^{k_1}, \tag{38}$$

$$p_n(z) = p_{n-1}(z)^{k_1}, \tag{39}$$

$$q_n(z) = \theta_{n-1}(z)^{k_1}, \tag{40}$$

whereas

$$\theta_n(z) = \delta_{n-1}(z)^{k_2}, \tag{41}$$

$$q_n(z) = \sigma_{n-1}(z)^{k_2}, \tag{42}$$

$$r_n(z) = r_{n-1}(z)^{k_2}. \tag{43}$$

By induction hypothesis, it follows:

$$\theta_{n-1}(z) = \delta_{n-2}^{c_1} \dots^{c_{n-3}}(x) = \sigma_{n-1}^{c_1}(x).$$

Therefore, from the expressions of q_n , it follows that $k_2 = c_1 k_1$. So, taking $c_{n-2} = k_1$ and using induction hypothesis to deduce p_{n-1} and r_{n-1} from p and $\{c_1, \dots, c_{n-3}\}$, the result that was intended to be proved is satisfied. Once the expression for the binary operators $\{R_n\}_{n>1}$ and $\{L_n\}_{n>1}$ in function of $\{c_n\}_{n \geq 1}$, $\{\delta_n\}_{n \geq 2}$, and p , has been obtained, the expression of ϕ_n is trivially inferred using induction on n by applying the recursive definition of ϕ_n . \square

The result established in the above Theorem 6.2 can be extended to the case when the connective rule is not necessarily standard.

Theorem 6.3. *Let*

$$\{\phi_n : [0, 1]^n \rightarrow [0, 1]\}_{n>1}$$

be a regular recursive rule, and let π be the underlying ordering rule. If $\phi_2 : [0, 1]^2 \rightarrow [0, 1]$ is strictly increasing in $(0, 1]^2$ and $\phi_2(0, x) = \phi_2(x, 0) = 0 \forall x \in [0, 1]^n$. Then, there exist

- (a) $p : [0, 1] \rightarrow [0, 1]$ an automorphism,
- (b) $\{\delta_n : [0, 1] \rightarrow [0, 1]\}_{n>1}$ automorphism family,
- (c) $\{c_n\}_{n \geq 1}$ a sequence of positive real numbers,

verifying

$$\phi_n(a_1, \dots, a_n) = \delta_n^{-1} \left(\left(\prod_{k=1}^n p(a_{\pi(k)})^{c_1^{k-1}} \right)^{\prod_{j=2}^{n-2} c_j} \right) \quad \forall (a_1, \dots, a_n) \in [0, 1]^n, \quad \forall n \geq 2,$$

where $\prod_{j=2}^{\ell} c_j$ with $\ell \leq 2$ is taken equal to 1.

Theorem 6.4. *Let*

$$\{\phi_n : [0, 1]^n \rightarrow [0, 1]\}_{n>1}$$

be a regular standard recursive rule. If $\phi_2 : [0, 1]^2 \rightarrow [0, 1]$ verifies the boundary conditions

- (i) $\phi_2(1, x) = \phi_2(x, 1) = 1 \forall x \in [0, 1]^n$,
- (ii) $\phi_2(x, 0) = \phi_2(0, x) \forall x \in [0, 1]^n$.

Then, there exist

- (a) $\psi : [0, 1] \rightarrow [0, 1]$ a strict negation,
- (b) $\{\delta_n : [0, 1] \rightarrow [0, 1]\}_{n>1}$ strict negation family,
- (c) $f^n : [0, 1]^n \rightarrow [0, 1]$, where f^n is defined from $f^2 = f$ just using associativity, being

$$f : [0, 1]^2 \rightarrow [0, 1]$$

and ordinal sum of strict t -norms.

Verifying, for all $n \geq 2$,

- $\phi_n(x_1, \dots, x_n) = \delta_n^{-1}(f^n(\psi(x_1), \dots, \psi(x_n))) \quad \forall (x_1, \dots, x_n) \in [0, 1]^n$,
- $L_n(a, b) = \delta_n^{-1}(f(\delta_{n-1}(a), \psi(b))) = R_n(b, a) \quad \forall (a, b) \in [0, 1]^2$.

Proof. Since $L_2 = \phi_2 = R_2$ and $L_2(1, x) = R_2(1, x) = R_2(x, 1) = L_2(x, 1) \quad \forall x \in [0, 1]$ the solution for equation

$$L_3(R_2(x_1, x_2), x_3) = R_3(x_1, L_2(x_2, x_3))$$

is

$$\begin{aligned} R_2(u, v) &= \sigma_3^{-1}(f(p_3(u), q_3(v))), \\ L_2(u, v) &= \theta_3^{-1}(f(q_3(u), r_3(v))), \\ L_3(a, b) &= \delta_3^{-1}(f(\sigma_3(a), r_3(b))), \\ R_3(a, b) &= \delta_3^{-1}(f(p_3(a), \theta_3(b))), \end{aligned} \tag{44}$$

given $\sigma_3, \theta_3, \delta_3, p_3, q_3, r_3$ are order reversing automorphisms and f is an ordinal sum of strict t -norms.

Therefore, following an analogous scheme as we did for Theorem 6.1, it follows that the solution for the generalized associativity equation for n ,

$$L_n(R_{n-1}(u, v), w) = R_n(u, L_{n-1}(v, w)),$$

would be

$$\begin{aligned} R_{n-1}(u, v) &= \delta_n^{-1}(f(\psi(u), \delta_{n-2}(v))), \\ L_{n-1}(u, v) &= \delta_n^{-1}(f(\delta_{n-2}(u), \psi(v))), \\ L_n(a, b) &= \delta_n^{-1}(f(\delta_{n-1}(a), \psi(b))), \\ R_n(a, b) &= \delta_n^{-1}(f(\psi(a), \delta_{n-1}(b))), \end{aligned}$$

those being dual solutions of the ones obtained in Theorem 6.1. \square

Theorem 6.5. Let

$$\{\phi_n : [0, 1]^n \rightarrow [0, 1]\}_{n>1}$$

be a regular standard recursive rule, verifying

- (i) $\phi_2(x, 1) = \phi_2(1, x) = 1 \quad \forall x \in [0, 1]^n$,
- (ii) ϕ_2 strictly increasing in $[0, 1] \times [0, 1]$.

Then there exist

- (a) $p : [0, 1] \rightarrow [0, 1]$ a strict negation,
- (b) $\{\delta_n\}_{n \geq 2}$ strict negations,

(c) $\{c_n\}_{n \geq 1}$ a sequence of positive real numbers, verifying

$$\phi_n(a_1, \dots, a_n) = \delta_n^{-1} \left(\left(\prod_{k=1}^n p(a_k)^{c_1^{k-1}} \right)^{\prod_{j=2}^{n-2} c_j} \right) \quad \forall (a_1, \dots, a_n) \in [0, 1]^n \quad \forall n \geq 2,$$

where $\prod_{j=2}^{\ell} c_j$ with $\ell \leq 2$ is taken equal to 1.

Proof. According to Mak [17], the solution for equation

$$L_3(R_2(x_1, x_2), x_3) = R_3(x_1, L_2(x_2, x_3))$$

would be

$$\begin{aligned} R_2(u, v) &= \tilde{\sigma}_3^{-1} \left(f(\tilde{p}_3(u), \tilde{q}_3(v)) \right), \\ L_2(u, v) &= \tilde{\theta}_3^{-1} \left(f(\tilde{q}_3(u), \tilde{r}_3(v)) \right), \\ L_3(a, b) &= \tilde{\delta}_3^{-1} \left(f(\tilde{\sigma}_3(a), \tilde{r}_3(b)) \right), \\ R_3(a, b) &= \tilde{\delta}_3^{-1} \left(f(\tilde{p}_3(a), \tilde{\theta}_3(b)) \right), \end{aligned}$$

given

$$\tilde{\sigma}_3, \tilde{\theta}_3, \tilde{\delta}_3, \tilde{p}_3, \tilde{q}_3, \tilde{r}_3$$

are order reversing automorphism and f an ordinal sum of strict t -norms. Obtaining the dual solutions of the ones we obtained in Theorem 6.2, just following the same reasoning we did there. \square

When the recursive rule is not standard, the above theorem can be extended as follows:

Theorem 6.6. *Let*

$$\{\phi_n : [0, 1]^n \rightarrow [0, 1]\}_{n > 1}$$

be a regular recursive rule, and let π be the underlying ordering rule, verifying

- (i) $\phi_2(x, 1) = \phi_2(1, x) = 1 \quad \forall x \in [0, 1]^n$,
- (ii) ϕ_2 strictly increasing in $[0, 1] \times [0, 1]$.

Then there exist

- (a) $p : [0, 1] \rightarrow [0, 1]$ a strict negation,
- (b) $\{\delta_n\}_{n \geq 2}$ strict negations,
- (c) $\{c_n\}_{n \geq 1}$ a sequence of positive real numbers,

verifying

$$\phi_n(a_1, \dots, a_n) = \delta_n^{-1} \left(\left(\prod_{k=1}^n p(a_{\pi(k)})^{c_1^{k-1}} \right)^{\prod_{j=2}^{n-2} c_j} \right) \quad \forall (a_1, \dots, a_n) \in [0, 1]^n, \quad \forall n \geq 2,$$

where $\prod_{j=2}^{\ell} c_j$ with $\ell \leq 2$ is taken equal 1.

7. Final comments

As shown in this paper, recursiveness allows to define an operative concept of *rule*. Some particular representation theorems have been obtained here, by taking advantage of Mak's solution for the general associativity equation [17]. Alternative axiomatic approaches have been proposed by other authors, sometimes based upon an *overly formal* link between connectives within a rule.

Main Theorem 3.1 and all other results in this paper show conditions leading to a basically additive solution. As a consequence, it is shown that if we impose the aggregation to be strictly increasing, then we easily can conclude that our aggregation is based upon weighted sums. Indeed, this is an extremely important result, with obvious consequences in any aggregation procedure. This is a quite standard situation in Multi-criteria and Group Decision making, Classification and Time Series, for example. Such a basically additive representation can be then assured, avoiding the restrictive associative approach which is based upon a unique binary operator.

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