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The equalizer and the lexicographical solutions for cooperative fuzzy games: characterization and properties

Elisenda Molina[a,*], Juan Tejada[b]

[a] Centro de Investigación Operativa, Dpto. de Estadística y Matemática Aplicada, Facultad de C.C. Experimentales, Universidad Miguel Hernández de Elche, 03202 Elche, Spain

[b] Dpto. de Estadística e I.O., Facultad de CC. Matemáticas, Universidad Complutense de Madrid, 28040 Madrid, Spain

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Abstract

In this paper we analyze the lexicographical solution for fuzzy TU games, we study its properties and obtain a characterization. The lexicographical solution was introduced by Sakawa and Nishizaki (Fuzzy Sets and Systems 61 (1994) 265 –275) as a solution for crisp TU games, and then extended as a value for fuzzy TU games. We approach the problem by means of the close relationship that exists between the lexicographical solution for crisp TU games and the least square nucleolus, a crisp value defined by Ruiz et al. (Internat. J. Game Theory 25 (1996) 113–134). Previously, and also based on this relationship, we axiomatically characterize the equalizer solution for fuzzy TU games. Both values, the equalizer and the lexicographical solutions, are based on the consideration of a measure of dissatisfaction of players rather than coalitions.

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1. Introduction

Cooperative games with fuzzy coalitions had been introduced by Aubin [1]. As in the nonfuzzy case, where only crisp coalitions are allowed to form, the study of solution concepts for games with fuzzy coalitions is a question of principal interest. Aubin [2,3] generalizes classical solution concepts such as the core and the Shapley value. Sakawa and Nishizaki [11] introduce a solution concept for crisp games, the lexicographical solution, and extend it to fuzzy games. The lexicographical solution for crisp games is a solution concept for TU games based on the lexicographical framework which deals with well-established solution concepts, such as the nucleolus [12] and the prenucleolus [15], but considers a measure of dissatisfaction of players rather

* Corresponding author. Tel.: +34-96-665-8696; fax: +34-96-665-8715.

E-mail address: e.molina@umh.es (E. Molina), jtejada@mat.ucm.es (J. Tejada).

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than coalitions. On the other hand, Ruiz et al. [10] propose another solution concept for crisp TU games, the least square nucleolus. The least square nucleolus, as the nucleolus, is based on the coalition excess vector. However, it differs in the assessment of the relative fairness with respect to that given by the lexicographical order. As a final payoff, it selects the imputation which minimizes the variance of the resulting excesses of coalitions over the imputation set. That is, both values are inspired on the nucleolus but each one takes a different feature into account. By extending the formulation of the least square nucleolus to fuzzy games, we prove that the lexicographical solution is the natural extension of this crisp value to fuzzy games.

Section 2 is devoted to a general presentation of notions and notations on a fuzzy game. In Section 3 we propose a new solution concept for fuzzy games, the equalizer solution, closely related to the lexicographical solution. We study its properties and give an axiomatic characterization. Finally, in Section 4 we give an alternative characterization of the lexicographical solution of Sakawa and Nishizaki, which turns out to be a better approach in order to analyze it. We study its properties and propose a polynomial algorithm to calculate it.

2. Cooperative games with fuzzy coalitions

An n-person cooperative game in characteristic function form with transferable utility (TU game) is an ordered pair (N, v), where N is a finite set of n players and $v : \mathscr{P}(N) \to \mathbb{R}$ is a map assigning a real number $v(S)$, called the value of S, to each coalition $S \subseteq N$, and where $v(\emptyset) = 0$. The real number $v(S)$ represents the reward that coalition S can achieve by itself if all its members act together.

Based on this concept, Aubin [1] defines a fuzzy game with transferable utility (TU fuzzy game) considering the concept of fuzzy coalition.

Definition 1 (Aubin). Let N be a set of n players. Then a *fuzzy coalition of N* is defined as a fuzzy subset of N.

Then, a fuzzy coalition is a function $\tau : N \to [0, 1]$, where $\tau(i)$ represents the rate of participation of player i in the fuzzy coalition τ. In this context, a fuzzy coalition τ is identified with the vector $\boldsymbol{\tau} \in [0, 1]^n$, where $\tau_i = \tau(i)$, for all $i \in N$.

The term fuzzy coalition arises when the possibility of graduating the membership of a player in a coalition is considered. Billot [5] describes a fuzzy coalition as a collection of economic agents, i.e., players, who transfer fractions of their representation to a collective decision maker, the fuzzy coalition.

Of course, in many real cooperative games only crisp coalitions are possible. However, in other cases the restriction to crisp coalitions is an excessive idealization. Butnariu [6] argues the necessity of fuzzy coalitions in political games for dealing with those situations in which a country cannot transfer all its decisional rights to a coalition but can be simultaneously a member in many coalitions. As an example, he gives the position of the United Kingdom "*which refuses to transfer its economical decisional rights to the West-European Parliament, but is a member of* E.E.C. *and of the Commonwealth in spite of their contradictory request in many questions*" ([6], p. 190).

Remark 1. Since a crisp coalition $S \subseteq N$ is a subset of N, it can be identified with its characteristic vector, $\boldsymbol{\tau}^S \in \{0, 1\}^n$, where $\tau_i^S = 1$, if $i \in S$, and $\tau_i^S = 0$, if $i \notin S$. Therefore, crisp coalitions are special cases of fuzzy coalitions. From now on, we will denote crisp coalitions by $S \subseteq N$.

Definition 2 (Aubin). A *TU game with fuzzy coalitions* is an ordered pair, $(N, \tilde{v})$, where N is a finite set of players and the characteristic function, $\tilde{v}$, is defined on its fuzzy subsets, i.e., $\tilde{v} : [0, 1]^n \to \mathbb{R}$, being $\tilde{v}(\mathbf{0}) = 0$.

In fact, the concept of TU game with fuzzy coalitions appeared in Shapley and Shubik [14], who spoke of players participating in a coalition at fractional levels of intensity (via "bundles of personal commodities").

One of the main topics dealt with in cooperative game theory is, given a game $(N, \tilde{v})$, to divide the amount $\tilde{v}(N)$ between players if the grand coalition N is formed. A payoff vector is any $\mathbf{x} \in \mathbb{R}^n$. A payoff vector is said to be efficient or a preimputation if $\sum_{i \in N} x_i = \tilde{v}(N)$ and an imputation if it is efficient and individually rational, i.e., $x_i \geqslant \tilde{v}(\{i\})$, for all $i \in N$. $PI(\tilde{v})$ denotes the set of preimputations; whereas $I(\tilde{v})$ the set of imputations.

For any payoff vector $\mathbf{x} \in \mathbb{R}^n$ and any fuzzy coalition $\boldsymbol{\tau} \neq \mathbf{0}$, the excess of $\boldsymbol{\tau}$ with respect to the payoff vector $\mathbf{x}$ is defined as $\tilde{e}(\boldsymbol{\tau}, \mathbf{x}) = \tilde{v}(\boldsymbol{\tau}) - \mathbf{x}\boldsymbol{\tau}$. Here, the scalar product $\mathbf{x}\boldsymbol{\tau}$ is the total amount that coalition $\boldsymbol{\tau}$ achieves according to the payoff vector $\mathbf{x}$, when it is assumed linearity on the distribution of payments. Then, the excess of a fuzzy coalition can be interpreted as a measure of dissatisfaction of coalition $\boldsymbol{\tau}$ if $\mathbf{x}$ were suggested as a final payoff.

Alternatively, Sakawa and Nishizaki [11] consider a measure of dissatisfaction of players. They define the excess of a player by means of the excesses of all coalitions which he/she belongs to. That is, let $\mathbf{x} \in \mathbb{R}^n$ be a payoff vector, then the excess of player i with respect to the payoff vector $\mathbf{x}$ is defined to be

$$\tilde{w}(i, \mathbf{x}) = \int_{[0,1]^n} \tau_i \tilde{e}(\boldsymbol{\tau}, \mathbf{x}) \, \mathrm{d}\boldsymbol{\tau}.$$

Consequently, each player can evaluate a payoff vector by means of all coalitions which he/she belongs to from a global point of view. By defining this measure of dissatisfaction, they make use of a lexicographical minimization procedure as the criteria to select the final payoff vector. For any payoff vector $\mathbf{x} \in \mathbb{R}^n$, let $\theta(\mathbf{x})$, which we will refer to as the excess vector, be the n-tupla whose components are the excesses $\tilde{w}(i, \mathbf{x})$, $i \in N$, arranged in a weakly decreasing order. Then, the lexicographical solution is defined as follows:

Definition 3 (Sakawa–Nishizaki [11]). The *lexicographical solution* of a TU fuzzy game, $(N, \tilde{v})$, denoted $L(\tilde{v})$, is the imputation which minimizes, according to the lexicographical order on $\mathbb{R}^n$, the excess vector, i.e.

$$\theta(L(\tilde{v})) \leqslant_L \theta(\mathbf{x}), \quad \forall \mathbf{x} \in I(\tilde{v}).$$

The involved excesses are usually nonnegative and therefore, the excesses are regarded as losses or complaints; whereas the excess vectors $\theta(\mathbf{x}) \in \mathbb{R}^n$ are interpreted as complaint vectors. Now the lexicographical order on $\mathbb{R}^n$ is used to order the excess vectors by taking their largest complaint into account or, if such should be the case, their second largest complaint and so on.

3. The equalizer solution

From now on, $F\Gamma^n$ will denote the class of all n-person fuzzy games with side payments and $F\Gamma_1^n$ the subclass of $F\Gamma^n$ composed by those games, $(N, \tilde{v})$, such that $\tilde{v} \in L^1(\mu)$, where μ is the Lebesgue measure on $[0,1]^n$, i.e., $\int_{[0,1]^n} |\tilde{v}| \, \mathrm{d}\mu < \infty$.

After introducing the lexicographical solution, Sakawa and Nishizaki [11] point out the interest of a solution equalizing the resulting excesses of all players and define the *solution which takes the same excess for all of players*. The problem of this solution being that its existence cannot be assured over the entire class $F\Gamma_1$, not even, over the subclass of $F\Gamma_1$ composed by those games with non-empty imputation set. We overcame this problem by enlarging the set of feasible payoff vectors to the set of preimputations.

Proposition 1. *For any game $(N, \tilde{v}) \in F\Gamma_1^n$ there exists a unique preimputation $\mathbf{x}$ verifying*

$$\tilde{w}(1, \mathbf{x}) = \tilde{w}(2, \mathbf{x}) = \cdots = \tilde{w}(n, \mathbf{x}). \tag{1}$$

It is given by

$$x_i = \frac{\tilde{v}(N)}{n} + 12(c_i(\tilde{v}) - \bar{c}(\tilde{v})), \quad i = 1,\dots,n, \tag{2}$$

where

$$c_i(\tilde{v}) = \int_{[0,1]^n} \tau_i \tilde{v}(\boldsymbol{\tau})\,\mathrm{d}\boldsymbol{\tau} \quad \textit{and} \quad \bar{c}(\tilde{v}) = \frac{\sum_{j=1}^n c_j(\tilde{v})}{n}.$$

Proof. For any efficient payoff, $\mathbf{x}$, the excess of player i with respect to $\mathbf{x}$ takes the following expression:

$$\tilde{w}(i,\mathbf{x}) = \int_{[0,1]^n} \tau_i \tilde{e}(\boldsymbol{\tau},\mathbf{x})\,\mathrm{d}\boldsymbol{\tau} = \int_{[0,1]^n} \tau_i \tilde{v}(\boldsymbol{\tau})\,\mathrm{d}\boldsymbol{\tau} - \frac{1}{3}x_i - \frac{1}{4}\sum_{j\neq i} x_j = c_i(\tilde{v}) - \frac{1}{12}x_i - \frac{1}{4}\tilde{v}(N). \tag{3}$$

Let $\mathbf{x}$ be a preimputation satisfying Eq. (1), then

$$x_i - x_j = 12(c_i(\tilde{v}) - c_j(\tilde{v})), \quad \forall i,j \in N.$$

Let us consider the constants $d^{ij} = 12c_i(\tilde{v}) - 12c_j(\tilde{v})$, for all $i,j \in N$. Then, the system of constants $\{d^{ij}\}_{i,j\in N}$ is compatible in the sense of Hart and Mas-Colell [8], i.e., $d^{ii} = 0$, $d^{ij} = -d^{ji}$ and $d^{ij} + d^{jk} = d^{ik}$, for all $i,j,k \in N$. Moreover, it follows from (3) that $\mathbf{x}$ has to preserve differences according to that system. Therefore, $\mathbf{x}$ exists, it is unique and is given by

$$x_i = \frac{1}{n}\left(\tilde{v}(N) + \sum_{j=1}^n d^{ij}\right), \quad x_j = x_i - d^{ij}.$$

Equivalently,

$$x_i = \frac{\tilde{v}(N)}{n} + \frac{12}{n}\left(nc_i(\tilde{v}) - \sum_{j=1}^n c_j(\tilde{v})\right), \quad i = 1,\dots,n. \qquad \square$$

Hart and Mas-Colell interpret the property of preserving differences as measuring the “relative position” (or “relative strengths”) of the players by means of the difference between $c_i(\tilde{v})$ and $c_j(\tilde{v})$.

Definition 4. Let $(N,\tilde{v}) \in F\Gamma_1^n$. Then, the *equalizer solution* for the game $(N,\tilde{v})$, denoted $\mathscr{E}(\tilde{v})$, is defined as the unique preimputation verifying Eq. (1).

Remark 2. Consider the game $(N,\tilde{v}) \in F\Gamma_1^n$. If its equalizer solution is individually rational, then it coincides with the lexicographical solution of the game (see Sakawa and Nishizaki [11]).

Remark 3. This property, to give out the global excess among the players in equal amounts, characterizes the *least square prenucleolus* for crisp TU games [10], which is defined as the unique preimputation which minimizes the variance of the resulting excesses of the coalitions over the preimputation set. For any TU game (N,v), its least square prenucleolus, denoted $\lambda(v)$, takes the following expression:

$$\lambda_i(v) = \frac{v(N)}{n} + \frac{1}{n2^{n-2}}\left(na_i(v) - \sum_{j\in N} a_j(v)\right), \quad i = 1,\dots,n,$$

where

$$a_i(v) = \sum_{\substack{S \subset N \\ i \in S}} v(S).$$

Thus, the equalizer solution extends the least square prenucleolus to fuzzy games by means of its alternative characterization. The following proposition shows that the equalizer solution extends this crisp value by means of its original definition too.

Consider the following problem for any fuzzy game $(N,\tilde{v}) \in F\Gamma_1^n$:

Problem 1:

$$\begin{aligned} \min \quad & \int_{[0,1]^n} (\tilde{e}(\boldsymbol{\tau},\mathbf{x}) - \bar{E}(\tilde{v},\mathbf{x}))^2 \, \mathrm{d}\boldsymbol{\tau} \\ \text{s.t.} \quad & \sum_{i \in N} x_i = \tilde{v}(N), \end{aligned}$$

where $\bar{E}(\tilde{v},\mathbf{x})$ is the average excess at $\mathbf{x}$, given by

$$\bar{E}(\tilde{v},\mathbf{x}) = \int_{[0,1]^n} \tilde{e}(\boldsymbol{\tau},\mathbf{x}) \, \mathrm{d}\boldsymbol{\tau}.$$

Lemma 1. *For any TU fuzzy game $(N,\tilde{v}) \in F\Gamma_1^n$, the average excess is the same for all efficient payoff vectors.*

Proof. Let $\mathbf{x}$ be any efficient payoff vector, then

$$\bar{E}(\tilde{v},\mathbf{x}) = \int_{[0,1]^n} \left(\tilde{v}(\boldsymbol{\tau}) - \sum_{i \in N} x_i \tau_i \right) \mathrm{d}\boldsymbol{\tau} = \int_{[0,1]^n} \tilde{v}(\boldsymbol{\tau}) \, \mathrm{d}\boldsymbol{\tau} - \frac{1}{2} \sum_{i \in N} x_i = \int_{[0,1]^n} \tilde{v}(\boldsymbol{\tau}) \, \mathrm{d}\boldsymbol{\tau} - \frac{1}{2} \tilde{v}(N),$$

which does not depend on $\mathbf{x}$. □

In the sequel we will denote the average excess at any efficient payoff vector by $\bar{E}(\tilde{v})$, instead of $\bar{E}(\tilde{v},\mathbf{x})$.

Proposition 2. *For any fuzzy game $(N,\tilde{v}) \in F\Gamma_1$, there exists a unique solution to Problem* 1 *and it is the equalizer solution.*

Proof. First, we will show that the Karush–Kuhn–Tucker conditions are necessary and sufficient conditions for global optimality for the following problem:

$$\begin{aligned} \min \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & h(\mathbf{x}) = 0, \\ & \mathbf{x} \in \mathbb{R}^n, \end{aligned}$$

where

$$f(\mathbf{x}) = \int_{[0,1]^n} (\tilde{e}(\boldsymbol{\tau},\mathbf{x}) - \bar{E}(\tilde{v}))^2 \, \mathrm{d}\boldsymbol{\tau} \quad \text{and} \quad h(\mathbf{x}) = \sum_{i \in N} x_i - \tilde{v}(N), \quad \forall \mathbf{x} \in \mathbb{R}^n.$$

In that case, it follows straightforward from the definition that f is strictly convex. Thus, since it is differentiable, it is pseudoconvex. Moreover, the unique constraint function is linear. Therefore, the Karush–Kuhn–

Tucker conditions are necessary and sufficient conditions for global optimality and $\mathbf{x} \in \mathbb{R}^n$ is the solution of the above problem if, and only if, there exists a scalar, λ, such that

$$\nabla f(\mathbf{x}) + \lambda \nabla h(\mathbf{x}) = \mathbf{0}, \tag{4}$$

$$h(\mathbf{x}) = 0. \tag{5}$$

Now we will show that $\mathscr{E}(\tilde{v})$ is the unique vector in $\mathbb{R}^n$ which satisfies the KKT conditions for that problem.

The characteristic function of the game is integrable in $[0,1]^n$. Then, f is differentiable with continuity and the partial derivative of f at $\mathbf{x}$ is given by

$$\frac{\partial f}{\partial x_i}(\mathbf{x}) = \int_{[0,1]^n} \frac{\partial(\tilde{e}(\boldsymbol{\tau},\mathbf{x}) - \bar{E}(\tilde{v}))^2}{\partial x_i}(\mathbf{x})\,\mathrm{d}\boldsymbol{\tau} = -2\tilde{w}(i,\mathbf{x}) + \bar{E}(\tilde{v}) \tag{6}$$

for all $i \in N$. Thus, Eq. (4) can be expressed as follows:

$$-2\tilde{w}(i,\mathbf{x}) + \bar{E}(\tilde{v}) + \lambda = 0, \quad \forall i = 1,\ldots,n. \tag{7}$$

It is deduced from expression (3) of the excess of a player with respect to any $\mathbf{x} \in \mathbb{R}^n$ verifying (5), i.e., efficient, that condition (7) above can be expressed as $-2c_i(\tilde{v}) + \frac{1}{6}x_i + \frac{1}{2}\tilde{v}(N) + \bar{E}(\tilde{v}) + \lambda = 0$, for all $i = 1,\ldots,n$. Then, $x_i = 12c_i(\tilde{v}) - 3\tilde{v}(N) - 6\bar{E}(\tilde{v}) - 6\lambda$, for all $i = 1,\ldots,n$. Thus, by efficiency, we get

$$6\lambda = 12\bar{c}(\tilde{v}) - \frac{3n+1}{n}\tilde{v}(N) - 6\bar{E}(\tilde{v}).$$

Therefore,

$$x_i = \frac{\tilde{v}(N)}{n} + 12(c_i(\tilde{v}) - \bar{c}(\tilde{v})) = \mathscr{E}_i(\tilde{v}), \quad \forall i = 1,\ldots,n. \qquad \square$$

After characterizing the equalizer solution as the unique preimputation which minimizes the variance of the resulting excesses of coalitions over the preimputation set, we study other properties.

Notation. For all vector $\boldsymbol{\tau} \in [0,1]^{n-1}$, and for every scalar $t \in [0,1]$, $(t_i, \boldsymbol{\tau})$ denotes the array $(\tau_1,\ldots,\tau_{i-1}, t, \tau_{i+1},\ldots,\tau_n) \in [0,1]^n$, for all $i = 1,\ldots,n$.

Definition 5. Let $(N,\tilde{v}) \in F\Gamma^n$ be a given TU fuzzy game. Player i is said to be a *dummy player* if, and only if,

$$\tilde{v}(t_i,\boldsymbol{\tau}) = \tilde{v}(0_i,\boldsymbol{\tau}) + t\tilde{v}(\{i\}), \quad \forall t \in [0,1], \ \forall \boldsymbol{\tau} \in [0,1]^{n-1}.$$

Definition 6. Let $(N,\tilde{v}) \in F\Gamma^n$ be a given TU fuzzy game. Players $i,j \in N$ are said to be *substitutes* if, and only if

$$\tilde{v}(t'_i, t''_j, \boldsymbol{\tau}) = \tilde{v}(t''_i, t'_j, \boldsymbol{\tau}), \quad \forall t', t'' \in [0,1], \ \forall \boldsymbol{\tau} \in [0,1]^{n-2}.$$

Definition 7. Two fuzzy games, $(N,\tilde{v})$ and $(N,\tilde{w})$, are said to be *strategically equivalent* if there exist a positive number $k > 0$ and n real constants $a_1,\ldots,a_n$ such that, for all $\boldsymbol{\tau} \in [0,1]^n$,

$$\tilde{w}(\boldsymbol{\tau}) = k\tilde{v}(\boldsymbol{\tau}) + \sum_{i \in N} a_i \tau_i.$$

Essentially, if two games are strategically equivalent, we can obtain one from the other simply by changing linearly the scale of measure and assigning a fixed benefit ($a_i > 0$) or cost ($a_i < 0$) associated to the presence of each player in a coalition. In that case, the fixed amount with which a player contributes to coalition $\boldsymbol{\tau}$ is proportional to his/her participation rate in that coalition.

Definition 8. A value on $F\Gamma_1^n$, $\varphi: F\Gamma_1^n \to \mathbb{R}^n$, satisfies the property of:

(i) *Equal treatment*: If $\varphi_i(\tilde{v}) = \varphi_j(\tilde{v})$, for every pair of substitutes i, j, and for all $(N, \tilde{v}) \in F\Gamma_1^n$.

(ii) *Anonymity*: Let $\pi: N \to N$ be any permutation of the set N of players. If we set

$$\pi\tilde{v}(\boldsymbol{\tau}) = \tilde{v}(\tau_{\pi(1)}, \ldots, \tau_{\pi(n)}), \quad \forall \boldsymbol{\tau} \in [0,1]^n$$

as the permuted game, $(N, \pi\tilde{v})$. Then

$$\varphi_i(\pi\tilde{v}) = \varphi_{\pi^{-1}(i)}(\tilde{v}), \quad \forall i \in N, \ \forall (N, \tilde{v}) \in F\Gamma_1^n.$$

(iii) *Additivity*: If for any pair of games $(N, \tilde{v})$, $(N, \tilde{w}) \in F\Gamma_1^n$,

$$\varphi_i(\tilde{v} + \tilde{w}) = \varphi_i(\tilde{v}) + \varphi_i(\tilde{w}), \quad \forall i \in N,$$

where the sum game $(N, \tilde{v} + \tilde{w})$ is defined as $(\tilde{v} + \tilde{w})(\boldsymbol{\tau}) = \tilde{v}(\boldsymbol{\tau}) + \tilde{w}(\boldsymbol{\tau})$, for all $\boldsymbol{\tau} \in [0,1]^n$.

(iv) *Inessential game*: If for any *inessential game* $(N, \tilde{v})$, that is $\tilde{v}(\boldsymbol{\tau}) = \sum_{i \in N} \tau_i \tilde{v}(\{i\})$, $\forall \boldsymbol{\tau} \in [0,1]^n$, holds

$$\varphi_i(\tilde{v}) = \tilde{v}(\{i\}), \quad \forall i \in N.$$

(v) *Strategic equivalence*: If for any pair of strategically equivalent games, $(N, \tilde{v})$, $(N, \tilde{w}) \in F\Gamma_1^n$,

$$\varphi_i(\tilde{w}) = k\varphi_i(\tilde{v}) + a_i, \quad \forall i \in N,$$

where k and $\mathbf{a}$ are the constants such that $\tilde{w}(\boldsymbol{\tau}) = k\tilde{v}(\boldsymbol{\tau}) + \sum_{i \in N} a_i \tau_i$, $\forall \boldsymbol{\tau} \in [0,1]^n$.

(vi) *Dummy player*: If for any dummy player $i \in N$ in a fuzzy game $(N, \tilde{v}) \in F\Gamma_1^n$, $\varphi_i(\tilde{v}) = \tilde{v}(\{i\})$.

(vii) *Average marginal contribution monotonicity*: If $c_i(\tilde{v}) \geqslant c_j(\tilde{v})$, then $\varphi_i(\tilde{v}) \geqslant \varphi_j(\tilde{v})$, for all $i, j \in N$, for all $(N, \tilde{v}) \in F\Gamma_1^n$.

(viii) *Coalitional monotonicity*: If for any set of players N, and any two characteristic functions $\tilde{v}$, $\tilde{w}$ on N, the existence of a subset of players $T \subseteq N$ for which conditions 1, 2, 3 and 4 below are fulfilled, implies $\varphi_i(\tilde{v}) \geqslant \varphi_i(\tilde{w})$, $\forall i \in T$.

1. $\tilde{v}(\boldsymbol{\tau}) = \tilde{w}(\boldsymbol{\tau})$, $\forall \boldsymbol{\tau} \in [0,1]^n$ such that $T \nsubseteq \mathrm{Sop}(\boldsymbol{\tau}) = \{i \in N / \tau_i > 0\}$.
2. $\tilde{v}(\boldsymbol{\tau}) \geqslant \tilde{w}(\boldsymbol{\tau})$, $\forall \boldsymbol{\tau} \in [0,1]^n$ such that $T \subseteq \mathrm{Sop}(\boldsymbol{\tau})$.
3. The difference game $(N, f) \in F\Gamma_1^n$, defined as $f(\boldsymbol{\tau}) = \tilde{v}(\boldsymbol{\tau}) - \tilde{w}(\boldsymbol{\tau})$, $\forall \boldsymbol{\tau} \in [0,1]^n$, is weakly increasing in members of T; whereas it is weakly decreasing in members of $N \backslash T$.
4. Players in T (respectively, in $N \backslash T$) are substitutes in the difference game.

(ix) *Strict coalitional monotonicity*: If the difference game above is strictly increasing in members of T; whereas it is strictly decreasing in members of $N \backslash T$, over $(0,1)^n$, then for all $i \in T$, $\varphi_i(\tilde{v}) > \varphi_i(\tilde{w})$.

(x) *Weak continuity*: If for any sequence of fuzzy games $\{(N, \tilde{v}_k)\}_{k \in \mathbb{N}}$ in $F\Gamma_1^n$, such that the sequence $\{\tilde{v}_k\}_{k \in \mathbb{N}}$ converges uniformly to $\tilde{v}_0$ holds $(N, \tilde{v}_0) \in F\Gamma_1^n$ and $\lim_{k \to \infty} \varphi(\tilde{v}_k) = \varphi(\tilde{v}_0)$.

Anonymity implies equal treatment and, obviously, strict coalitional monotonicity implies coalitional monotonicity.

Remark 4. The inequality $c_i(\tilde{v}) \geqslant c_j(\tilde{v})$ can be equivalently expressed as

$$\int_{\{\tau_i > \tau_j\}} (\tau_i - \tau_j)\tilde{v}(\boldsymbol{\tau})\, \mathrm{d}\boldsymbol{\tau} \geqslant \int_{\{\tau_j > \tau_i\}} (\tau_j - \tau_i)\tilde{v}(\boldsymbol{\tau})\, \mathrm{d}\boldsymbol{\tau}.$$

Then, property (vii) can be interpreted as follows: Let us consider the weighted aggregated value of those coalitions in which player i's participation is greater than player j's participation. If this value is not less than the weighted aggregated value of those coalitions in which, on the contrary, it is player j who participates more, then i should not receive less than j.

Coalitional monotonicity properties are the fuzzy extension of those crisp monotonicity properties introduced by Young [17]. Those properties establish that if a group of players cooperate to make an investment to develop a more efficient project, which implies an improvement of their productivity, then that group should not be penalized. For instance, let $\alpha > 0$ be the profit increment generated by the improvement in the production process of T members, and let

$$f(\boldsymbol{\tau}) = \prod_{i \in T} \tau_i \prod_{i \notin T} (1 - \tau_i)\alpha, \quad \forall \boldsymbol{\tau} \in [0,1]^n$$

be the characteristic function of the difference game. Then, the profit increment is expanded in a multilinear way to other coalitions which are different from T. Efficiency of fuzzy coalition $\boldsymbol{\tau}$ production process increases proportionally to, on the one hand, the presence of T members and, on the other, the absence of $N \backslash T$ members.

Proposition 3. *The equalizer solution*, $\mathscr{E} : F\Gamma_1^n \to \mathbb{R}^n$, *verifies properties* (i) *to* (x), *but not* (vi).

Proof. Properties (i), (ii), (iii), (iv), (v) and (vii) can easily be deduced from expression (2) of the equalizer solution obtained in Proposition 1.

With respect to continuity, let $\{(N, \tilde{v}_k)\}_{k \in \mathbb{N}}$ be any sequence in $F\Gamma_1^n$. If $\{\tilde{v}_k\}_{k \in \mathbb{N}}$ converges uniformly to $\tilde{v}_0$, then $\tilde{v}_0 \in L^1(\mu)$ and, for each $i \in N$, holds

$$\lim_{k \to \infty} c_i(\tilde{v}_k) = \lim_{k \to \infty} \int_{[0,1]^n} \tau_i \tilde{v}_k(\boldsymbol{\tau}) \, \mathrm{d}\boldsymbol{\tau} = \int_{[0,1]^n} \lim_{k \to \infty} \tau_i \tilde{v}_k(\boldsymbol{\tau}) \, \mathrm{d}\boldsymbol{\tau} = \int_{[0,1]^n} \tau_i \tilde{v}_0(\boldsymbol{\tau}) \, \mathrm{d}\boldsymbol{\tau} = c_i(\tilde{v}_0).$$

Since $\lim_{k \to \infty} \tilde{v}_k(N) = \tilde{v}_0(N)$, for each $i \in N$, holds

$$\lim_{k \to \infty} \mathscr{E}_i(\tilde{v}_k) = \lim_{k \to \infty} \left(\frac{\tilde{v}_k(N)}{n} + 12(c_i(\tilde{v}_k) - \bar{c}(\tilde{v}_k)) \right) = \frac{\tilde{v}_0(N)}{n} + 12(c_i(\tilde{v}_0) - \bar{c}(\tilde{v}_0)) = \mathscr{E}_i(\tilde{v}_0).$$

Now, we will prove that the equalizer solution is strict coalitional monotonic. Let $(N, \tilde{v})$, $(N, \tilde{w})$ be any two games for which conditions 1, 2, 3 and 4 hold. Then, by additivity, $\mathscr{E}(\tilde{v}) = \mathscr{E}(\tilde{w}) + \mathscr{E}(f)$.

Given that players in T (respectively, in $N \backslash T$) are substitutes in the difference game (N, f), by anonymity, we have

$$\mathscr{E}_i(\tilde{v}) = \mathscr{E}_i(\tilde{w}) + \frac{f(N)}{n} + 12(a_1 - \bar{a}), \quad \forall i \in T,$$

$$\mathscr{E}_i(\tilde{v}) = \mathscr{E}_i(\tilde{w}) + \frac{f(N)}{n} + 12(a_2 - \bar{a}), \quad \forall i \notin T,$$

where $a_1 = c_i(f)$, $\forall i \in T$, $a_2 = c_i(f)$, $\forall i \notin T$, and $\bar{a} = (ta_1 + (n-t)a_2)/n$, being $t = |T|$. Since $f(N) > 0$, if $T = N$ then

$$\mathscr{E}_i(\tilde{v}) = \mathscr{E}_i(\tilde{w}) + \frac{f(N)}{n} > \mathscr{E}_i(\tilde{w}), \quad \forall i \in N.$$

Otherwise, $T \subsetneq N$, by proving $a_1 > \bar{a}$, or equivalently $a_1 > a_2$, the inequality $\mathscr{E}_i(\tilde{v}) > \mathscr{E}_i(\tilde{w})$ follows.

For each pair of players $i \in T$, $j \notin T$, by the monotonicity of the difference game, holds

$$f(t_i', t_j'', \boldsymbol{\tau}) \geqslant f(t_i'', t_j'', \boldsymbol{\tau}) \geqslant f(t_i'', t_j', \boldsymbol{\tau}), \quad \forall t' \geqslant t'' \text{ and } \forall \boldsymbol{\tau} \in [0,1]^{n-2},$$

whereas

$$f(t_i', t_j'', \boldsymbol{\tau}) > f(t_i'', t_j'', \boldsymbol{\tau}) > f(t_i'', t_j', \boldsymbol{\tau}), \quad \forall 1 > t' > t'' > 0 \text{ and } \forall \boldsymbol{\tau} \in (0,1)^{n-2}.$$

Then,

$$\int_{\{\tau_i > \tau_j\}} (\tau_i - \tau_j) f(\boldsymbol{\tau})\, \mathrm{d}\boldsymbol{\tau} > \int_{\{\tau_i < \tau_j\}} (\tau_j - \tau_i) f(\boldsymbol{\tau})\, \mathrm{d}\boldsymbol{\tau}$$

and, therefore, $a_1 > a_2$. □

Now we will prove the coincidence between the least square prenucleolus [10] of a crisp game and the equalizer solution of the fuzzy game defined by its multilinear extension [9]. It gives us, on the one hand, a method for calculating the least square prenucleolus of a crisp game by means of its multilinear extension, and, on the other, it enables us to show that the equalizer solution does not satisfy all those properties which are not satisfied by the least square prenucleolus.

Proposition 4. *Let (N, v) be a given crisp game. Consider the TU fuzzy game $(N, \tilde{v})$ defined by its multilinear extension. Then, $\lambda(v) = \mathscr{E}(\tilde{v})$, where $\lambda(v)$ is the least square prenucleolus of (N, v).*

Proof. The fuzzy game defined by the multilinear extension of (N, v) is given by

$$\tilde{v}(\boldsymbol{\tau}) = \sum_{S \subset N} \prod_{i \in S} \tau_i \prod_{i \notin S} (1 - \tau_i) v(S), \quad \forall \boldsymbol{\tau} \in [0,1]^n.$$

Thus, each $c_i(\tilde{v})$, $i = 1, \ldots, n$, can be expressed as follows:

$$\begin{aligned} c_i(\tilde{v}) &= \int_{[0,1]^n} \tau_i \tilde{v}(\boldsymbol{\tau})\, \mathrm{d}\boldsymbol{\tau} = \sum_{S \subset N} \int_{[0,1]^n} \tau_i \prod_{j \in S} \tau_j \prod_{j \notin S} (1 - \tau_j) v(S)\, \mathrm{d}\boldsymbol{\tau} \\ &= \frac{1}{3 \cdot 2^{n-1}} \sum_{\substack{S \subset N \\ i \in S}} v(S) + \frac{1}{3 \cdot 2^n} \sum_{\substack{S \subset N \\ i \notin S}} v(S) = \frac{1}{3 \cdot 2^n} a_i(v) + \frac{1}{3 \cdot 2^n} \sum_{S \subset N} v(S), \end{aligned} \tag{8}$$

where

$$a_i(v) = \sum_{\substack{S \subset N \\ i \in S}} v(S).$$

Then

$$\bar{c}(\tilde{v}) = \frac{1}{3n \cdot 2^n} \sum_{j \in N} a_j(v) + \frac{1}{3 \cdot 2^n} \sum_{S \subset N} v(S).$$

Therefore

$$\mathscr{E}_i(\tilde{v}) = \frac{\tilde{v}(N)}{n} + 12(c_i(\tilde{v}) - \bar{c}(\tilde{v})) = \frac{v(N)}{n} + \frac{1}{n 2^{n-2}} \left(n a_i(v) - \sum_{j \in N} a_j(v) \right), \quad i = 1, \ldots, n,$$

which is the explicit expression of the least square prenucleolus of (N, v). □

Remark 5. It follows from the previous proposition that:
1. The equalizer solution fails to verify the dummy axiom.
2. The equalizer solution of a game with non-empty core does not always belong to its core.

Certainly, let $(N,\tilde{v})$ be a multilinear fuzzy game. If $i\in N$ is a dummy player in $(N,\tilde{v})$, so she/he is in the crisp game (N,v). Therefore, since Ruiz et al. [10] prove that least square prenucleolus fails to verify the dummy axiom, claim 1 holds. With respect to claim 2, as the core of $(N,\tilde{v})$ equals the core of (N,v) (see Tejada [16]), it is deduced from the fact that least square prenucleolus verifies strict coalitional monotonicity, which implies that least square prenucleolus of a balanced game does not always belong to its core (see Ruiz et al. [10]).

Now, we offer an axiomatization involving axioms which are usual in the characterization of certain solution concepts plus a new axiom: to satisfy the property of average marginal contribution monotonicity.

Theorem 1. *The equalizer solution is the unique value on* $F\Gamma_1^n$, $\varphi: F\Gamma_1^n\to\mathbb{R}^n$, *which verifies the following axioms*:
A1. *Efficiency.*
A2. *Inessential game.*
A3. *Additivity.*
A4. *Average marginal contribution monotonicity.*

Proof. It has been proved in Proposition 3 that the equalizer solution, which is by its definition efficient, verifies A2–A4. Now, we will show that it is the unique value on $F\Gamma_1^n$ satisfying A1–A4.

Let φ be a value on $F\Gamma_1^n$ which satisfies A1–A4. Let $(N,\tilde{v})$ be a game in $F\Gamma_1^n$. We will prove that $\varphi(\tilde{v})\in PI(\tilde{v})$ preserves differences according to the system of compatible constants $\{d^{ij}\}$ previously defined.

For any pair of distinct players $i,j\in N$, let $(N,\tilde{v}_{ij})$ be the game defined as the sum of the original game, $(N,\tilde{v})$, and the inessential game, (N,w_{ij}^*), which is given by

$$w_{ij}^*(\boldsymbol{\tau}) = a_i\tau_i + a_j\tau_j, \quad \forall\boldsymbol{\tau}\in[0,1]^n,$$

where

$$a_i = -\tfrac{36}{7}(c_i(\tilde{v})-c_j(\tilde{v})) \quad\text{and}\quad a_j = \tfrac{48}{7}(c_i(\tilde{v})-c_j(\tilde{v})).$$

As (N,w_{ij}^*) is an inessential game, by axiom A2, $\varphi_i(w_{ij}^*)=a_i$ and $\varphi_j(w_{ij}^*)=a_j$. Additivity implies

$$\varphi_i(\tilde{v}_{ij}) = \varphi_i(\tilde{v}) + \varphi_i(w_{ij}^*) = \varphi_i(\tilde{v}) - \tfrac{36}{7}(c_i(\tilde{v})-c_j(\tilde{v})), \tag{9}$$

$$\varphi_j(\tilde{v}_{ij}) = \varphi_j(\tilde{v}) + \varphi_j(w_{ij}^*) = \varphi_j(\tilde{v}) + \tfrac{48}{7}(c_i(\tilde{v})-c_j(\tilde{v})). \tag{10}$$

Since $c_i(\tilde{v}_{ij})=c_i(\tilde{v})=c_j(\tilde{v}_{ij})$, by axiom A4, $\varphi_i(\tilde{v}_{ij})=\varphi_j(\tilde{v}_{ij})$. Therefore, it follows from expressions (9) and (10), that $\varphi_i(\tilde{v})-\varphi_j(\tilde{v})=12(c_i(\tilde{v})-c_j(\tilde{v}))$, for all $i,j\in N$. Given that $\mathscr{E}(\tilde{v})$ is the unique preimputation which preserves differences according to that system of constants, $\mathscr{E}(\tilde{v})=\varphi(\tilde{v})$. □

Proposition 5. *The axioms* A1–A4 *are logically independent.*

Proof. To show the independence of these four axioms we will prove that for every group of three axioms there exists a solution which satisfies them all except the fourth one.

($\neg$A1) Let φ^1 be the value on $F\Gamma_1^n$ defined as

$$\varphi_i^1(\tilde{v}) = \int_{[0,1]^n} 6\tau_i(\tilde{v}(\boldsymbol{\tau}) - \tilde{v}((1-\tau_i)_i,\boldsymbol{\tau}))\,\mathrm{d}\boldsymbol{\tau}, \quad \forall i\in N,\ \forall (N,\tilde{v})\in F\Gamma_1^n,$$

which can be expressed as

$$\varphi_i^1(\tilde{v}) = 6\left(2c_i(\tilde{v}) - \int_{[0,1]^n} \tilde{v}(\boldsymbol{\tau})\,\mathrm{d}\boldsymbol{\tau}\right), \quad \forall i \in N,\ \forall (N,\tilde{v}) \in F\Gamma_1^n. \tag{11}$$

Then, it follows from (11) that φ^1 satisfies A2–A4. Now, we will show that φ^1 is not efficient in general. Let $(N,\tilde{v})$ be a multilinear fuzzy game, i.e., a fuzzy game defined as the multilinear extension of a certain crisp game (N,v). Then, by replacing in expression (11) the value of $c_i(\tilde{v})$, $i \in N$, by that one obtained in (8), and taking into account that

$$\int_{[0,1]^n} \tilde{v}(\boldsymbol{\tau})\,\mathrm{d}\boldsymbol{\tau} = \frac{1}{2^n}\sum_{S\subseteq N} v(S)$$

for each player $i \in N$, holds

$$\varphi_i^1(\tilde{v}) = \frac{1}{2^{n-1}}\left(\sum_{\substack{S\subseteq N\\ i\in S}} v(S) - \sum_{\substack{S\subseteq N\\ i\notin S}} v(S)\right) = \frac{1}{2^{n-1}}\sum_{\substack{S\subseteq N\\ i\in S}} (v(S) - v(S\backslash\{i\})) = \beta_i(v).$$

Thus, $\varphi^1(\tilde{v}) = \beta(v)$, where $\beta(v)$ is the Banzhaf–Coleman value (Banzhaf [4], Coleman [7]) of (N,v). Therefore, as $\tilde{v}(N) = v(N)$ and the Banzhaf–Coleman value does not verify efficiency, neither does φ^1.

($\neg$A2) The value in $F\Gamma_1^n$, φ^2, defined as

$$\varphi_i^2(\tilde{v}) = \frac{\tilde{v}(N)}{n} + (c_i(\tilde{v}) - \bar{c}(\tilde{v})), \quad \forall i \in N,\ \forall (N,\tilde{v}) \in F\Gamma_1^n \tag{12}$$

satisfies all axioms except inessential game.

($\neg$A3) Let φ^3 be the value on $F\Gamma_1^n$ which assigns to each $(N,\tilde{v}) \in F\Gamma_1^n$, the payoff vector obtained at the end of the following algorithm:

Step 1: Consider the set $K = \{j \in N / \mathscr{E}_j(\tilde{v}) < \tilde{v}(\{j\})\}$. If $K = \emptyset$, then $\varphi^3(\tilde{v}) := \mathscr{E}(\tilde{v})$. Stop. Otherwise, go to Step 2.

Step 2: Let $j_0 \in K$ be the player such that $\mathscr{E}_{j_0}(\tilde{v}) = \min_{j\in K} \mathscr{E}_j(\tilde{v})$.
Consider the partition of the player set, S, T, given by

$$T := \{i \in N\backslash K / c_i(\tilde{v}) < c_{j_0}(\tilde{v})\},$$
$$S := \{i \in N / c_i(\tilde{v}) \geqslant c_{j_0}(\tilde{v})\}.$$

If $T = \emptyset$, then $\varphi^3(\tilde{v}) := \mathscr{E}(\tilde{v})$. Stop.
Otherwise, go to Step 3.

Step 3:

$$\varphi_j^3(\tilde{v}) := \begin{cases} \mathscr{E}_j(\tilde{v}) - \dfrac{s\delta}{t} & \text{if } j \in T, \\ \mathscr{E}_j(\tilde{v}) + \delta & \text{if } j \in S, \end{cases}$$

where $\delta = \tilde{v}(\{j_0\}) - \mathscr{E}_{j_0}(\tilde{v}) > 0$, $s = |S|$ and $t = |T|$. Stop.

φ^3 verifies all axioms except additivity. Let $(N,\tilde{v}_1)$ and $(N,\tilde{v}_2)$ be the *TU* fuzzy games defined by $N = \{1,2,3\}$ and

$$\tilde{v}_1(\boldsymbol{\tau}) = \tau_1\tau_2\tau_3 + \tau_1\tau_3 - \tau_2\tau_3 + \tau_2, \quad \forall \boldsymbol{\tau} \in [0,1]^n,$$

$$\tilde{v}_2(\boldsymbol{\tau}) = \tau_1\tau_3 + 2\tau_1\tau_2 + 2\tau_2\tau_3 - \tau_2 - 2\tau_1\tau_2\tau_3, \quad \forall \boldsymbol{\tau} \in [0,1]^n.$$

Then,

	$\mathscr{E}(\tilde{v})$	K	T	S	δ	$\varphi^3(\tilde{v})$
$\tilde{v}_1$	$(\frac{5}{6},\frac{5}{6},\frac{1}{3})$	$K=\{2\}$	$\{3\}$	$\{1,2\}$	$\frac{1}{6}$	$(1,1,0)$
$\tilde{v}_2$	$(\frac{5}{6},\frac{1}{3},\frac{5}{6})$	$\emptyset$				$(\frac{5}{6},\frac{1}{3},\frac{5}{6})$
$\tilde{v}_1+\tilde{v}_2$	$(\frac{5}{3},\frac{7}{6},\frac{7}{6})$	$\emptyset$				$(\frac{5}{3},\frac{7}{6},\frac{7}{6})$

($\neg$A4) Let φ^4 be the value on $F\Gamma_1^n$ defined as the Shapley value [13] of the crisp game, (N, v), given by the restriction of $\tilde{v}$ to crisp coalitions, i.e.

$$\varphi_i^4(\tilde{v}) = \sum_{\substack{S \subset N \\ i \in S}} \frac{(s-1)!(n-s)!}{n!} (\tilde{v}(\mathbf{x}^S) - \tilde{v}(\mathbf{x}^{S\setminus\{i\}})) \quad \forall i \in N, \ \forall (N,\tilde{v}) \in F\Gamma_1^n.$$

Obviously, φ^4 satisfies A1–A3. With respect to A.M.C. monotonicity, let $(N,\tilde{v})$ be a fuzzy multilinear game. Then, it follows from (8) that

$$c_i(\tilde{v}) \geqslant c_j(\tilde{v}) \ \Leftrightarrow \ a_i(v) \geqslant a_j(v).$$

Thus, since the Shapley value fails to verify crisp A.M.C. monotonicity [1] (see Ruiz et al. [10]), so does φ^4. □

4. The lexicographical solution

In this section we analyze the lexicographical solution [11] as the extension of the least square nucleolus [10] to fuzzy games. First, we will prove that the lexicographical solution admits a characterization in terms of the excesses of coalitions, rather than players, which is analogous to that which defines the least square nucleolus. This approach will enable us to offer a polynomial algorithm to calculate the lexicographical solution of a fuzzy game by means of its equalizer solution, for which we have obtained a simple analytical expression.

Formally, consider the following problem for any fuzzy game $(N,\tilde{v}) \in F\Gamma_1^n$:

Problem 2:

$$\begin{aligned} \min \quad & \int_{[0,1]^n} (\tilde{e}(\boldsymbol{\tau},\mathbf{x}) - \bar{E}(\tilde{v}))^2 \, d\boldsymbol{\tau} \\ \text{s.t.} \quad & \sum_{i\in N} x_i = \tilde{v}(N), \\ & x_i \geqslant \tilde{v}(\{i\}), \quad i = 1,\ldots,n. \end{aligned}$$

In order to characterize the lexicographical solution as the unique solution of Problem 2, we first establish the following.

[1] If $a_i(v) \geqslant a_j(v)$, then player i should not receive less than player j.

Lemma 2. *For any fuzzy game* $(N,\tilde{v})\in F\Gamma_1^n$ *with non-empty imputation set there exists a unique solution of Problem 2. It is characterized as imputation* $\mathbf{z}$ *satisfying for all* $j\in N$,

$$z_j>\tilde{v}(\{j\})\Rightarrow \tilde{w}(j,\mathbf{z})=\max_{i\in N}\tilde{w}(i,\mathbf{z}). \tag{13}$$

Proof. The feasible set of Problem 2, the set of imputations, is compact. The objective function f, given by

$$f(\mathbf{x})=\int_{[0,1]^n}(\tilde{e}(\boldsymbol{\tau},\mathbf{x})-\bar{E}(\tilde{v}))^2\,\mathrm{d}\boldsymbol{\tau},\quad \forall\mathbf{x}\in\mathbb{R}^n$$

is strictly convex. Thus, the minimum always exists and it is unique.

Obviously, if $I(\tilde{v})=\{(\tilde{v}(\{1\}),\dots,\tilde{v}(\{n\}))\}$, then $(\tilde{v}(\{1\}),\dots,\tilde{v}(\{n\}))$, which trivially fulfills condition (13), is the unique solution of Problem 2.

Otherwise, in order to prove that the unique solution of Problem 2 is the unique imputation which fulfills condition (13) for every player $j\in N$, first, we will show that the Karush–Kuhn–Tucker conditions are necessary and sufficient conditions for global optimality for the following problem, whenever $I(\tilde{v})\neq\{(\tilde{v}(\{1\}),\dots,\tilde{v}(\{n\}))\}$.

$$\begin{aligned}
\min\quad & f(\mathbf{x})\\
\text{s.t.}\quad & g_i(\mathbf{x})\leqslant 0,\quad i=1,\dots,n,\\
& h(\mathbf{x})=0,\\
& \mathbf{x}\in\mathbb{R}^n,
\end{aligned}$$

where f is as previously defined, $g_i(\mathbf{x})=\tilde{v}(\{i\})-x_i$, $i=1,\dots,n$, and $h(\mathbf{x})=\sum_{i\in N}x_i-\tilde{v}(N)$.

(a) f is differentiable and the constraint functions are linear. Moreover, $\nabla g_i(\mathbf{x})=(-1_i,\mathbf{0})$, for all $\mathbf{x}\in\mathbb{R}^n$, $i=1,\dots,n$, and $\nabla h(\mathbf{x})=\mathbf{1}$, $\forall\mathbf{x}\in\mathbb{R}^n$. Then, the vectors $\{\nabla h(\mathbf{x}),\nabla g_i(\mathbf{x}),\, i\in I\}$, $I=\{i/g_i(\mathbf{x})=0\}$, are linearly independent whenever $I(\tilde{v})\neq\{(\tilde{v}(\{1\}),\dots,\tilde{v}(\{n\}))\}$.

(b) The objective function is pseudoconvex and the constraint functions are quasiconvex and quasiconcave for all $\mathbf{x}\in\mathbb{R}^n$.

Therefore, the Karush–Kuhn–Tucker conditions are necessary and sufficient conditions for global optimality. Then, $\mathbf{z}\in\mathbb{R}^n$ is the solution of the problem above if, and only if, there exist scalars $v\in\mathbb{R}$, $u_1,\dots,u_n$, $u_i\geqslant 0$, $\forall i=1,\dots,n$, such that

$$\nabla f(\mathbf{z})+\sum_{i=1}^n u_i\nabla g_i(\mathbf{z})+v\nabla h(\mathbf{z})=\mathbf{0}, \tag{14}$$

$$u_i g_i(\mathbf{z})=0,\quad i=1,\dots,n. \tag{15}$$

It follows from (6) that the Karush–Kuhn–Tucker conditions can be expressed as follows:

$$-2\tilde{w}(i,\mathbf{z})+\bar{E}(\tilde{v})-u_i+v=0,\quad i=1,\dots,n, \tag{16}$$

$$u_i g_i(\mathbf{z})=0,\quad i=1,\dots,n. \tag{17}$$

Then, it follows from (17) that $u_j=0$, for every $j\in N$ with $z_j>\tilde{v}(\{j\})$. Therefore, it follows from (16), that

$$\tilde{w}(j,\mathbf{z})=\frac{\bar{E}(\tilde{v})+v}{2}.$$

Since $u_i \geqslant 0$, $\forall i = 1, \ldots, n$, by (16), we have

$$\tilde{w}(i, \mathbf{z}) = \frac{\bar{E}(\tilde{v}) + v - u_i}{2} \leqslant \frac{\bar{E}(\tilde{v}) + v}{2} = \tilde{w}(j, \mathbf{z}), \quad \forall i = 1, \ldots, n.$$

Thus, the unique solution of Karush–Kuhn–Tucker equations is an imputation which satisfies condition (13), for all $j \in N$. In particular, there exists an imputation in such conditions.

Now, we will prove that if $\mathbf{z} \in I(\tilde{v})$ fulfills condition (13), then it solves Karush–Kuhn–Tucker equations. Let us consider

$$v = 2M - \bar{E}(\tilde{v}),$$

$$u_i = 2(M - \tilde{w}(i, \mathbf{z})), \quad i = 1, \ldots, n,$$

where $M = \max_{i \in N} \tilde{w}(i, \mathbf{z})$. Obviously, $u_i \geqslant 0$, $\forall i = 1, \ldots, n$, $v \in \mathbb{R}$ and Eqs. (16) and (17) are fulfilled. □

Proposition 6. *For any TU fuzzy game, $(N, \tilde{v}) \in F\Gamma_1^n$, with non-empty imputation set, its lexicographical solution is the unique solution to Problem 2.*

Proof. First of all, we will prove that for any fuzzy game, $(N, \tilde{v}) \in F\Gamma_1^n$, the sum of the excesses of all players is the same for all efficient payoff vectors. Let $\mathbf{x}$ be any efficient payoff vector, then it follows from expression (3) that

$$\sum_{i \in N} \tilde{w}(i, \mathbf{x}) = \sum_{i \in N} c_i(\tilde{v}) - \frac{1}{12} \sum_{i \in N} x_i - \frac{n}{4} \tilde{v}(N) = \sum_{i \in N} c_i(v) - \frac{1 + 3n}{12} \tilde{v}(N) = \beta.$$

Let $\mathbf{z}$ be the solution to Problem 2, which exists and is unique according to Lemma 2. Consider the set $M = \{j \in N / z_j = v(\{j\})\}$. Then, by Lemma 2, $\tilde{w}(j, \mathbf{z}) = \tilde{w}(k, \mathbf{z})$, for all $j, k \notin M$. Therefore,

$$\tilde{w}(j, \mathbf{z}) = \frac{\beta - \bar{\beta}}{n - m}, \quad \forall j \notin M,$$

where $\bar{\beta} = \sum_{i \in M} \tilde{w}(i, \mathbf{z})$ and $m = |M|$.

Now, we shall show that any imputation, $\mathbf{x}$, such that $\theta(\mathbf{x}) \leqslant_L \theta(\mathbf{z})$, satisfies $\tilde{w}(i, \mathbf{x}) = \tilde{w}(i, \mathbf{z})$, for all $i \in N$.

(a) First, we prove that if $\theta(\mathbf{x}) \leqslant_L \theta(\mathbf{z})$, then $\tilde{w}(j, \mathbf{x}) = \tilde{w}(j, \mathbf{z})$, for all $j \notin M$. Let $\mathbf{x}$ be any imputation, then it follows from expression (3) of $\tilde{w}(i, \mathbf{x})$ that $\tilde{w}(i, \mathbf{x}) \leqslant \tilde{w}(i, \mathbf{z})$, for all $i \in M$. Therefore,

$$\sum_{i \in M} \tilde{w}(i, \mathbf{x}) \leqslant \sum_{i \in M} \tilde{w}(i, \mathbf{z}) = \bar{\beta}.$$

Thus

$$\sum_{i \notin M} \tilde{w}(i, \mathbf{x}) \geqslant \beta - \bar{\beta}. \tag{18}$$

By contradiction, suppose that $\tilde{w}(j, \mathbf{x}) < (\beta - \bar{\beta})/(n - m)$ for some player $j \notin M$. Then, it follows from (18) that there exists $\ell \neq j$, $\ell \notin M$, such that $\tilde{w}(\ell, \mathbf{x}) > (\beta - \bar{\beta})/(n - m)$. Hence

$$\theta_1(\mathbf{x}) = \max_{i \in N} \tilde{w}(i, \mathbf{x}) > \frac{\beta - \bar{\beta}}{n - m} = \max_{i \in N} \tilde{w}(i, \mathbf{z}) = \theta_1(\mathbf{z}).$$

That is, $\theta(\mathbf{x}) >_L \theta(\mathbf{z})$. Therefore, if $\mathbf{x}$ is an imputation such that $\theta(\mathbf{x}) \leqslant_L \theta(\mathbf{z})$, then

$$\tilde{w}(j,\mathbf{x}) \geqslant \frac{\beta - \bar{\beta}}{n-m}, \quad \forall j \notin M.$$

On the other hand, $\theta(\mathbf{x}) \leqslant_L \theta(\mathbf{z})$ implies

$$\max_{i \in N} \tilde{w}(i,\mathbf{x}) \leqslant \max_{i \in N} \tilde{w}(i,\mathbf{z}) = \frac{\beta - \bar{\beta}}{n-m}.$$

So, $\tilde{w}(j,\mathbf{x}) = (\beta - \bar{\beta})/(n-m) = \tilde{w}(j,\mathbf{z})$, for all $j \notin M$.

(b) Now, we prove that if $\mathbf{x}$ is an imputation such that $\theta(\mathbf{x}) \leqslant_L \theta(\mathbf{z})$, then $\tilde{w}(i,\mathbf{x}) = \tilde{w}(i,\mathbf{z})$, for all $i \in M$. Let $\mathbf{x}$ be an imputation such that $\theta(\mathbf{x}) \leqslant_L \theta(\mathbf{z})$, then it follows from (a) that $\tilde{w}(j,\mathbf{x}) = (\beta - \bar{\beta})/(n-m)$ for all $j \notin M$. Therefore,

$$\sum_{i \in M} \tilde{w}(i,\mathbf{x}) = \bar{\beta} = \sum_{i \in M} \tilde{w}(i,\mathbf{z}).$$

Furthermore, $\tilde{w}(i,\mathbf{x}) \leqslant \tilde{w}(i,\mathbf{z})$, for all $i \in M$. Hence, $\tilde{w}(i,\mathbf{x}) = \tilde{w}(i,\mathbf{z})$, for all $i \in M$.

It is deduced from (a) and (b) that $\tilde{w}(i,\mathbf{x}) = \tilde{w}(i,\mathbf{z}), \forall i \in N$, whenever $\theta(\mathbf{x}) \leqslant_L \theta(\mathbf{z})$. Then, it follows from expression (3) that $\mathbf{x} = \mathbf{z}$. Therefore, the solution to Problem 2 is the lexicographical minimum. □

Remark 6. The least square nucleolus admits a characterization analogous to that given in Lemma 2 (see Ruiz et al. [10]). Then, by adapting Proposition 6 to crisp games, it follows that the least square nucleolus can be characterized as the unique imputation which minimizes, according to the lexicographical order, the players' excess vector.

Lemma 2 offers an alternative characterization of the lexicographical solution which makes use of the following algorithm, proposed by Ruiz et al. [10] as a method to calculate the least square nucleolus, valid to calculate the lexicographical solution starting from the equalizer solution.

Algorithm (Ruiz et al. [10]). Construct a sequence of pairs $(\mathbf{x}^\ell, M^\ell)$, $\ell = 1,\dots,k$, $1 \leqslant k \leqslant n$, where $\mathbf{x}^\ell$ is a payoff vector and M^ℓ a subset of N, inductively defined by

Step 1: $\mathbf{x}^1 := \mathscr{E}(\tilde{v})$, $M^1 := \{j \in N / \mathscr{E}_j(\tilde{v}) < \tilde{v}(\{j\})\}$ and $\ell = 1$, where $\mathscr{E}(\tilde{v})$ is the equalizer solution of the game.

Step 2: $$x_j^{\ell+1} := \begin{cases} x_j^\ell + \dfrac{\sum_{j \in M^\ell}(x_j^\ell - \tilde{v}(\{j\}))}{n - m_\ell} & \text{for all } j \notin M^\ell, \\ \tilde{v}(\{j\}) & \text{for all } j \in M^\ell \end{cases}$$

and $M^{\ell+1} := M^\ell \cup \{j \in N / x_j^{\ell+1} < \tilde{v}(\{j\})\}$.

Step 3: If $M^{\ell+1} = M^\ell$, then $\mathbf{z} := \mathbf{x}^{\ell+1}$. Stop.
Otherwise, let $\ell := \ell + 1$ and go to Step 2.

Obviously, this process must end after at most $n-1$ steps and the closing payoff vector $\mathbf{z}$ is an imputation satisfying condition (13).

Next proposition, which shows how to obtain the least square nucleolus of a crisp game by means of its multilinear extension [9], supports the lexicographical solution as the natural extension to fuzzy games of such value.

Proposition 7. *Let (N,v) be a given crisp game. Consider the TU fuzzy game $(N,\tilde{v})$ defined by its multilinear extension. Then, $\Lambda(v)=L(\tilde{v})$, where $\Lambda(v)$ is the least square nucleolus of (N,v).*

Proof. The excess of each player $i\in N$ with respect to any imputation $\mathbf{x}$ in the fuzzy game $(N,\tilde{v})$ can be expressed in terms of the corresponding excess in the crisp game as follows:

It follows from expressions (3) and (8) that

$$\tilde{w}(i,\mathbf{x}) = \frac{1}{3\cdot 2^n}a_i(v) - \frac{1}{12}x_i - \frac{1}{4}v(N) + \frac{1}{3\cdot 2^n}\sum_{S\subseteq N} v(S).$$

Thus

$$3\cdot 2^n\tilde{w}(i,\mathbf{x}) = a_i(v) - 2^{n-2}x_i - 2^{n-2}\tilde{v}(N) - 2^{n-1}\tilde{v}(N) + \sum_{S\subseteq N} v(S).$$

On the other hand, the excess of player $i\in N$ with respect to $\mathbf{x}\in PI(v)=PI(\tilde{v})$ in the crisp game (N,v) is given by

$$w(i,\mathbf{x}) = \sum_{\substack{S\subseteq N\\ i\in S}} e(S,\mathbf{x}) = a_i(v) - 2^{n-1}x_i - 2^{n-2}\sum_{j\neq i} x_j = a_i(v) - 2^{n-2}x_i - 2^{n-2}v(N).$$

Therefore, $\tilde{w}(i,\mathbf{x})=kw(i,\mathbf{x})+a$, for all $\mathbf{x}\in I(v)=I(\tilde{v})$, where k and a are defined as

$$k = \frac{1}{3\cdot 2^n} > 0,$$

$$a = \frac{\sum_{S\subseteq N} v(S) - 2^{n-1}v(N)}{3\cdot 2^n}.$$

Thus, the excess of each player in the fuzzy game is obtained as a positive linear transformation of the corresponding excess in the crisp game. Then, since the imputation set is the same for both games, the result holds. □

We end up showing which of the least square nucleolus properties keep being fulfilled by the lexicographical solution.

Proposition 8. *The lexicographical solution, $L: IF\Gamma_1^n\to\mathbb{R}^n$, where $IF\Gamma_1^n$ is the subset of $F\Gamma_1^n$ composed by those games in $F\Gamma_1^n$ which have non-empty imputation set, verifies the following properties*: (i) *Equal treatment*, (ii) *Anonymity*, (iv) *Inessential game*, (v) *Strategic equivalence and* (x) *Weak continuity.*

Proof. Properties (i), (ii), (iv) and (v) can easily be deduced from the definition of the lexicographical solution by using the expression of the excess of a player given at the beginning of this section.

With respect to continuity, let $\{(N,\tilde{v}_r)\}_{r\in\mathbb{N}}$ be any sequence of fuzzy games in $IF\Gamma_1^n$. If the sequence $(\tilde{v}_r)_{r\in\mathbb{N}}$ converges uniformly to $\tilde{v}_0$ and $L(\tilde{v}_r)\to L_0$ as $r\to\infty$, then $\tilde{v}_0\in L^1(\mu)$ and $I(\tilde{v}_0)\neq\emptyset$, i.e., $\tilde{v}_0\in IF\Gamma_1^n$.

For each $i\in N$ holds

$$\lim_{r\to\infty}\tilde{w}_r(i,L(\tilde{v}_r)) = \lim_{r\to\infty}\int_{[0,1]^n}\tau_i\left(\tilde{v}_r(\boldsymbol{\tau}) - \sum_{j\in N}\tau_j L_j(\tilde{v}^r)\right)\mathrm{d}\boldsymbol{\tau}$$
$$= \int_{[0,1]^n}\tau_i\tilde{v}_0(\boldsymbol{\tau})\,\mathrm{d}\boldsymbol{\tau} - \frac{1}{3}L_i(\tilde{v}_0) + \frac{1}{4}\sum_{j\neq i} L_j(\tilde{v}_0) = \tilde{w}_0(i,L_0).$$

Let us show that $L_0 \in I(\tilde{v}_0)$ fulfills condition (13) established in Lemma 2.

Let $i \in N$ be a player such that $L_{0i} > \tilde{v}_0(\{i\})$. The sequences $(L_i(\tilde{v}_r))_{r\in\mathbb{N}}$ and $(\tilde{v}_r(\{i\}))_{r\in\mathbb{N}}$ converge to L_{0i} and $\tilde{v}_0(\{i\})$, respectively, as r approaches to ∞. Therefore, if we set

$$\mathscr{P} = \{r \in \mathbb{N} / L_i(\tilde{v}_r) > \tilde{v}_r(\{i\})\}$$

then $|\mathscr{P}| = \infty$. Thus,

$$\lim_{\substack{r\to\infty \\ r\in\mathscr{P}}} L(\tilde{v}_r) = L_0, \tag{19}$$

$$\lim_{\substack{r\to\infty \\ r\in\mathscr{P}}} \tilde{w}_r(j, L(\tilde{v}_r)) = \tilde{w}_0(j, L_0), \quad \forall j \in N. \tag{20}$$

On the other hand, since $L(\tilde{v}_r)$ is the lexicographical solution of $(N, \tilde{v}_r)$, we get

$$\max_{j\in N} \tilde{w}_r(j, L(\tilde{v}_r)) = \tilde{w}_r(i, L(\tilde{v}_r)), \quad \forall r \in \mathscr{P}$$

then, by the continuity of the maximum function, it follows from (20) that

$$\tilde{w}_0(i, L_0) = \lim_{\substack{r\to\infty \\ r\in\mathscr{P}}} \tilde{w}_r(i, L(\tilde{v}_r)) = \lim_{\substack{r\to\infty \\ r\in\mathscr{P}}} \max_{j\in N} \tilde{w}_r(j, L(\tilde{v}_r)) = \max_{j\in N} \tilde{w}_0(j, L_0).$$

Thus, L_0 fulfills condition (13), which characterizes the lexicographical solution. □

Assuring individual rationality causes the lost of additivity as well as A.M.C. monotonicity (Example 1). However, if we only consider (0, 1)-normalized games monotonicity can be guaranteed.

Example 1. Consider the fuzzy game $(N, \tilde{v})$ with 3 players and characteristic function

$$\tilde{v}(\boldsymbol{\tau}) = \tau_2 - \tau_2\tau_3 + \tau_1\tau_3 + \tau_1\tau_2\tau_3.$$

Then, $c_1(\tilde{v}) = \frac{3}{8}$, $c_2(\tilde{v}) = \frac{3}{8}$ and $c_3(\tilde{v}) = \frac{1}{3}$. Thus, the equalizer solution is $\mathscr{E}(\tilde{v}) = (\frac{5}{6}, \frac{5}{6}, \frac{1}{3})$. Since $\mathscr{E}_2(\tilde{v}) = \frac{5}{6} < \tilde{v}(\{2\})$, then, according to the algorithm, we get $L(\tilde{v}) = (\frac{3}{4}, 1, \frac{1}{4})$. As $c_1(\tilde{v}) = c_2(\tilde{v})$ the lexicographical solution does not satisfy A.M.C. monotonicity property.

Proposition 9. *Let* $\widetilde{F\Gamma}_1^n = \{(N, \tilde{v}) \in IF\Gamma_1^n / \tilde{v}(\{i\}) = \tilde{v}(\{j\})\ \forall i, j \in N\}$. *If we keep to* $\widetilde{F\Gamma}_1^n$, *then the lexicographical solution verifies A.M.C. monotonicity.*

Proof. The proof follows from the application of the previous algorithm. Given a fuzzy game $(N, \tilde{v}) \in \widetilde{F\Gamma}_1^n$, let $i, j \in N$ be any two players. We can assume without loss of generality that $c_i(\tilde{v}) \geqslant c_j(\tilde{v})$. Then, we will show that $L_i(\tilde{v}) \geqslant L_j(\tilde{v})$.

If the equalizer solution of the game, $\mathscr{E}(\tilde{v})$, is individually rational, then $L(\tilde{v}) = \mathscr{E}(\tilde{v})$. Therefore, as the equalizer solution verifies the monotonicity property, $L_i(\tilde{v}) = \mathscr{E}_i(\tilde{v}) \geqslant \mathscr{E}_j(\tilde{v}) = L_j(\tilde{v})$. Otherwise, two cases are possible:

1. If $\mathscr{E}_i(\tilde{v}) < \tilde{v}(\{i\})$, then $\tilde{v}(\{j\}) = \tilde{v}(\{i\}) > \mathscr{E}_i(\tilde{v}) \geqslant \mathscr{E}_j(\tilde{v})$. Thus

$$L_i(\tilde{v}) = \tilde{v}(\{i\}) = \tilde{v}(\{j\}) = L_j(\tilde{v}).$$

2. If $\mathscr{E}_i(\tilde{v}) \geqslant \tilde{v}(\{i\})$, then we have to consider the following cases:
   (a) If $\mathscr{E}_j(\tilde{v}) < \tilde{v}(\{j\})$, then $L_j(\tilde{v}) = \tilde{v}(\{j\}) = \tilde{v}(\{i\}) \leqslant L_i(\tilde{v})$.
   (b) If $\mathscr{E}_j(\tilde{v}) \geqslant \tilde{v}(\{j\})$, then there exist $p_i \geqslant 0$ and $p_j \geqslant 0$ such that

$$L_i(\tilde{v}) = \mathscr{E}_i(\tilde{v}) - p_i,$$

$$L_j(\tilde{v}) = \mathscr{E}_j(\tilde{v}) - p_j.$$

Since $\mathscr{E}_i(\tilde{v}) \geqslant \mathscr{E}_j(\tilde{v})$ and $\tilde{v}(\{i\}) = \tilde{v}(\{j\})$, then, according to the algorithm, we get $p_i \geqslant p_j$. Moreover, it holds:

$$\text{If } p_j < p_i \Rightarrow L_j(\tilde{v}) = \tilde{v}(\{j\}) = \tilde{v}(\{i\}) \leqslant L_i(\tilde{v}),$$

$$\text{If } p_j = p_i \Rightarrow L_j(\tilde{v}) = \mathscr{E}_j(\tilde{v}) - p_i \leqslant \mathscr{E}_i(\tilde{v}) - p_i = L_i(\tilde{v}). \qquad \square$$

If we take the $(0,1)$-normalization of the game given in Example 1

$$\tilde{v}_{(0,1)}(\boldsymbol{\tau}) = -\tau_2\tau_3 + \tau_1\tau_3 + \tau_1\tau_2\tau_3,$$

then $L(\tilde{v}_{(0,1)}) = L(\tilde{v}) - (0,1,0) = (\frac{3}{4}, 0, \frac{1}{4})$, which does not contradict the monotonicity with respect to average marginal contributions condition since the normalization operation does not respect the relative order among the contribution coefficients of the players. In that case, we have

$$c_1(\tilde{v}_{(0,1)}) = c_1(\tilde{v}) - \int_{[0,1]^3} \tau_1\tau_2 \, \mathrm{d}\boldsymbol{\tau} = \frac{1}{8} > \frac{1}{24} = c_2(\tilde{v}) - \int_{[0,1]^3} \tau_2^2 \, \mathrm{d}\boldsymbol{\tau} = c_2(\tilde{v}_{(0,1)}).$$

5. Concluding remarks

In this paper, we have analyzed some aspects of the lexicographical solution for fuzzy games by adopting a coalition excess approach. The approach followed by Sakawa and Nishizaki [11] shows the idea behind this value more clearly, but the formulation of Ruiz et al. [10] is the one to be used for obtaining good results. According to this reasoning, we have extended the formulation of the least square nucleolus to fuzzy games dealing with an alternative characterization of the lexicographical solution. Then, we have given an algorithm to calculate the lexicographical solution which is an effective method of calculus. Moreover, we have introduced a new value for fuzzy games with side payments, the equalizer solution, which shows good behaviour as a solution concept.

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