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An Integrated Transport System for Alacant's Students. UNIVERCITY

JOAQUÍN SÁNCHEZ-SORIANO, NATIVIDAD LLORCA, ANA MECA, ELISENDA MOLINA and MANUEL PULIDO
Dpto. Estadística y Matemática Aplicada and C.I.O., Universidad Miguel Hernández de Elche, Avda del Ferrocarril s.n., 03202 Elche (Alicante), España

Abstract. The dispersion between the different university campuses in *Alacant* raises the social necessity of designing a transport system capable of efficiently connecting the villages and cities of *Alacant* with the campuses. In this paper, we develop a centralized transport system for university students in the province of *Alacant*.

Keywords: cooperative game theory, applications, transportation networks, sharing cost rules

2000 AMS subject classification: 91A12, 91A80

1. Introduction

The university campuses in *Alacant* are widely dispersed. In addition, the social structure in Spain is focused on the family as a cellular unit and the majority of university students live with their parents while they study. The public transport system in this province does not cover the students' needs and, moreover, it is expensive. Therefore there is a real social need for a transport system capable of efficiently connecting the villages and cities of *Alacant* with the different university campuses. But at the same time, taking into account that the university system in Spain is mainly state controlled, such a system should also be economical. The *Comunitat Valenciana* government is aware of this problem and it has established a grants mechanism to reduce the fees and to encourage private firms to offer the transport service which society is demanding. This transport grants system is such that Student Unions (which have been formed to obtain cheap services), town councils and private firms can all apply for financial support. At present, all these entities fail to cooperate with one another. Yet there are many advantages in changing this situation and promoting an atmosphere of cooperation. One of the most important parameters to be taken into account in the financial support system is the number of students who apply for grants in each city. Then, if all town councils of *Alacant* apply for financial support, they will all together receive a total amount of financial support which would exceed what they are now receiving. For instance, during the last academic period some cites did not receive any financial support because no institution (town councils, Student Unions or private firms) applied. In more than half of the towns only private firms applied. Since, in this case, the rules for the grants specify that the maximum number of students taken into account will be a percentage of the student Census, the financial support for the remaining percentage was lost.

On this basis, an analysis of this problem has been carried out in order to make a proposal to the *Diputació d'Alacant*, where all town councils are represented. This will be the intermediary body between the students (through their corresponding town councils) and the *Generalitat Valenciana* for organising an integrated university transport system in the whole province. We indeed propose that the management of the transport system should be taken over by the *Diputació d'Alacant*. There would be many advantages in centralizing the management of such a transport network into only one public institution. In fact, a joint transport system managed by the *Diputació* would allow:

(1) the design of an optimal network (the best routes);

(2) an increase in the financial support obtained from the *Generalitat Valenciana*;

(3) a low fare for the bus service, since the negotiating position of the students, represented by the *Diputació*, would be strengthened;

(4) a reduction in traffic problems, because the students would not need to drive their own cars.

Moreover, if the new transport system turns out to be efficient and cheap, the *Diputació d'Alacant* by implication will receive political gains. Furthermore, a good service for the students could have positive effects by reducing family expenses. On the other hand, new trends in town planning consider that this kind of transport system would allow universities to integrate into cities. This is the so-called "UniverCity". Hence, cooperation clearly is worthwhile and, furthermore, there is an agent (the *Diputació*) who will be able to centralize the management of the whole transport system.

In order to design such a centralized transport system for university students who live and study in the province of *Alacant*, we consider three different steps: firstly, obtaining and analyzing data, secondly, designing optimal routes and schedules, and thirdly, once the optimal transport system has been designed, evaluating all the costs arising from the use of that system. Finally, these costs must be distributed among the users of the system, the students. This last stage is precisely the aim of this paper.

The problem we face, setting the fees for using the system, is a cost sharing problem. We will analise the problem in two contexts. In any case, prior to allocating the costs, we will divide the full network into several independent transport subsystems.

In section 3, we will focus on the cost sharing problem within the context of the cooperative game theory. Application of game theory to cost allocation problems is very common nowadays. Examples of cost sharing problems that are worked out in this setting are numerous, see for instance [3,8,13,14,21,22]. We will consider the town councils as the agents of the game and we will define the corresponding stand alone cost game (see [27,33], among others), which we refer to as *tree buses game played by the cities*. Then we will propose to share the total costs of the transport subsystem according to rules such as the Egalitarian Nonseparable Cost (ENSC) method or the Alternate Cost

Avoid (ACA) method (see [5,33]). Obviously, once the costs have been allocated to the cities, the cost share for each city is distributed equally among its students.

This approach, however, does not permit us to take into account all the special features of this problem. Therefore we have decided to resolve this situation by means of an alternative approach, which is considered in section 4. We will propose a cost sharing rule which embraces the principles of fairness that every public institution should respect. On this basis, we will first distinguish two types of costs, and will then propose a particular method to allocate them among the users. We will refer to this specific method as the *aggregated egalitarian solution*. In particular, the different agents involved in the decision making process, the students as the users, and the *Diputació* as the agent who must set the fees, have different interests. A thorough study of this fact has lead us to consider a compensatory monetary system. To be exact, we will propose the financial support to be covered by the *Diputació*.

Section 5 contains the conclusions and future research of our work.

2. An integrated transport system

As mentioned above, we must accomplish three different analyses in order to design the integrated transport system. The game theory will be used in the last stage of our study. Once the optimal transport system has been designed, we must evaluate all the costs resulting from the hiring and use of that system. Then the cost of the system is to be distributed among its users.

There are two types of costs to be distinguished, the *fixed costs* and the *variable costs*: the *fixed costs* are related to the costs paid per bus, no matter the distance covered. In many cases the unit hire fees depend on the amount x of buses that are hired. Generally, the unit price is reduced according to the number of buses contracted, although hiring $x+1$ buses is usually more expensive than hiring x. That is, if $f(x)$ represents the unit price per bus, when $x \in \mathbb{Z}_+$ buses are hired, then

$$f(x) \geqslant f(x+1) \quad \text{and} \quad xf(x) < (x+1)f(x+1).$$

The *variable costs* are prices paid per kilometer travelled by a bus. Among others, these costs include the maintenance costs of the buses. Therefore, if $c(k)$ represents the variable cost of hiring a bus for travelling a total distance of k kilometres (including the return trip), then the total cost of contracting $\mathbf{x} = (x_1, \dots, x_t)$ buses which are going to cover distances of $\mathbf{k} = (k_1, \dots, k_t)$ kilometres, respectively, is given by

$$H(\mathbf{x}, \mathbf{k}) = \left(\sum_{j=1}^{t} x_j\right) f\left(\sum_{j=1}^{t} x_j\right) + \sum_{j=1}^{t} x_j c(k_j). \tag{1}$$

In the situation we deal with, the number of buses hired will probably exceed the number which qualify for the lowest (highest) price, making it impossible to apply further price reductions (or rises), due to decreasing (increasing) returns to scale, when a large number of buses are hired (when long distances are covered). Hence, we will work on the particular case of constant fixed cost ($f \equiv F$) and linear variable cost ($c(k) = K \cdot k$). Then, it follows

$$H(\mathbf{x}, \mathbf{k}) = \left(\sum_{j=1}^{t} x_j \right) F + \left(\sum_{j=1}^{t} x_j k_j \right) K, \tag{2}$$

where F is the fixed cost of hiring a bus and K is the variable cost paid per kilometer covered.

Even though the transport system as a whole is too large, in practice it is made up of several independent transport subsystems. We will consider different subsystems for each destination, i.e., for each University Campus. Moreover, for each subsystem, we will distinguish between the routes by means of their area origins. We will put two routes together if, and only if, they converge on the final stages of their journey. Thus, the original problem will be simplified to allocate the costs of each independent subsystem, which we will refer to as *trees of routes with common sections*, among its users. Note that we can restrict ourselves to trees without loss of generality, since graphs with cycles cannot be optimal routes in our situation. In fact, if there were a cycle we would remove a link obtaining a new subgraph of shorter length.

Example 1. The buses covering the routes connecting the area of *Marina Baixa* with the *Universidad Miguel Hernández*, campus of *Sant Joan*, and the *Universitat d'Alacant* will be considered as the *Marina Baixa* area–*Alacant* transport subsystem. Graphically, the tree with common sections which represents that subsystem is in figure 1. The vertexes represent the cities:

Vertex 1	Vertex 2	Vertex 3	Vertex 4	Vertex 5
L'Alfàs del Pi	Calpe	Altea	Benidorm	La Vila Joiosa

s_j is the number of students of city j, $j = 1, \ldots, 5$, d_j is the distance (measured in kilometers) from city j to the next city on the route, and k_j is the distance from city j to the university campus: $k_1 = d_1 + d_4 + d_5$ and $k_i = \sum_{\ell=i}^{5} d_\ell$, $i = 2, \ldots, 5$.

Now, we should allocate the total costs of the *Marina Baixa* transport subsystem among university students who live in that area and take a bus from these routes to arrive

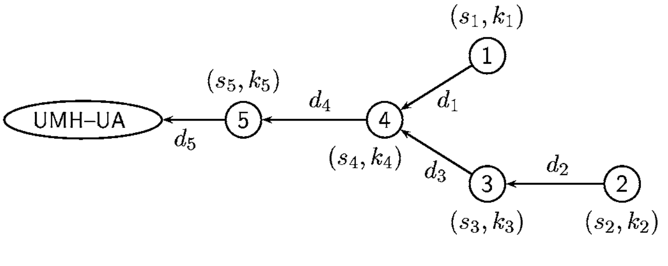

Figure 1. Marina Baixa subsystem.

at *Sant Joan*. That is, we should distribute the total costs $H(\mathbf{x}^*, \mathbf{k})$ given by (2), where $\mathbf{x}^* = (x_1^*, \dots, x_5^*)$ is the optimal fleet of buses. Here, x_i represents the number of buses which depart from city i, $i = 1, \dots, 5$, and $\mathbf{x}^*$ is the most economical fleet of buses, which will be determined by:

$$\begin{aligned} H(\mathbf{x}^*, \mathbf{k}) = \min\ & H(\mathbf{x}, \mathbf{k}) \\ \text{s.t.}\ & bx_1 \geqslant s_1, \\ & bx_2 \geqslant s_2, \\ & b(x_2 + x_3) \geqslant s_2 + s_3, \\ & b(x_1 + x_2 + x_3 + x_4) \geqslant s_1 + s_2 + s_3 + s_4, \\ & b(x_1 + x_2 + x_3 + x_4 + x_5) \geqslant s_1 + s_2 + s_3 + s_4 + s_5, \\ & x_i \in \mathbb{Z}_+, \quad i = 1, \dots, 5, \end{aligned} \tag{3}$$

where $b \in \mathbb{Z}_+$ is the capacity of the buses.

The cost sharing problem we deal with, which we will refer to as a *tree buses situation*, can formally be described by means of the 5-tuple $\mathcal{B} := (\mathrm{T}, \mathbf{s}, \mathbf{d}, b, H)$, where $\mathrm{T} = (V, E)$ is a directed tree (V is the set of vertices – cities and the campus terminus – and E is the set of edges – sections), the elements of $\mathbf{s} \in \mathbb{Z}_+^{\overline{V}}$ (where $\overline{V} = V \setminus \{\text{campus terminus}\}$) represent the number of students using the transport subsystem in each city, $\mathbf{d} \in \mathbb{R}_+^E$ is the vector of distances, $b \in \mathbb{Z}_+$ is the capacity of the buses (we will assume all buses have the same capacity, to simplify the situation), and H is the cost function. Here, $H(\mathbf{x}, \mathbf{k})$ represents the cost of hiring and making use of a fleet of x_i, $i \in \overline{V}$, buses which are covering total distances of k_i kilometres, $i \in \overline{V}$, respectively.

Now, let $\mathcal{B} = (\mathrm{T}, \mathbf{s}, \mathbf{d}, b, H)$ be a tree buses situation. Let $\mathbf{x}^*$ be the cheapest fleet of buses covering the routes of $\mathcal{B}$. Then we will approach the problem of allocating the total costs $H(\mathbf{x}^*, \mathbf{k})$ within two contexts. The first approach is analyzed in the next section. The second is postponed to section 4.

3. Cooperative game theoretical approach: tree buses game played by the cities

In our first attempt to distribute the cost of each subsystem we have fallen back on the cooperative game theory. This approach will make sense if the agents involved in the situation we have focused on, are able and willing to make binding agreements on the individual cost shares. It is debatable whether the students are qualified to make binding agreements. However, it is likely that the representatives of the town councils have the authority to make such agreements. Thus, we have decided to consider the cities as the players of the game. We then define the corresponding stand alone cost game, which is the usual approach in this setting.

The stand alone cost game, which we will refer to as *tree buses game played by the cities*, formally is given by the pair $(\overline{V}, c_{\mathcal{B}})$, where $c_{\mathcal{B}}(S)$ is defined as the minimum quantity needed to hire and use buses for taking all the students living in any city of coalition S to the campus, for all nonempty coalition $S \subseteq \overline{V}$, while $c_{\mathcal{B}}(\emptyset) = 0$. So, given a tree buses situation $\mathcal{B} = (\mathrm{T}, \mathbf{s}, \mathbf{d}, b, H)$, $(\overline{V}, c_{\mathcal{B}})$ refers to the hypothetical situation where each coalition $S \subseteq \overline{V}$ is associated with the cost of serving them first. Thus, the cost of each nonempty coalition $S \subseteq \overline{V}$ is obtained by means of solving the following optimization problem:

$$c_{\mathcal{B}}(S) = \min_{\mathbf{x}_S \in \mathbb{Z}_+^S} \{H(\mathbf{x}_S, \mathbf{k}_S) \mid b\mathbf{x}_S^t \mathbf{R}^S \geqslant \mathbf{s}_S^t \mathbf{R}^S\}, \tag{4}$$

where $\mathbf{s}_S = (s_i)_{i \in S}$, $\mathbf{x}_S = (x_i)_{i \in S}$, $\mathbf{k}_S = (k_i)_{i \in S}$, and $\mathbf{R}^S$ is the following square matrix of size $|S|$:

$$R_{ij}^S = \begin{cases} 1, & \text{if } i = j \text{ or } j \in P(i), \\ 0, & \text{otherwise,} \end{cases} \quad \forall i, j \in S.$$

$P(i) \subseteq \overline{V}$ denotes the set of cities in the unique path in the tree T connecting city i to the root of the tree, the campus.

Note that in the *Marina Baixa* example, if b is the capacity of the buses, then the optimal fleet $\mathbf{x}^*$ previously determined coincides with the optimal one obtained when solving the optimization problem (4) for $S = \overline{V}$. Moreover, the optimal fleet obtained as the solution of the previous problem is the same for every "reasonable" cost function, i.e., for any function that is nondecreasing in each variable. In such a way, the optimal solution can be defined recursively. First, we must assign the buses which depart from the cities located at each leaf of the tree. We must assign to each of these cities the minimum number of buses needed to move their students. Then, recursively, we complete the following cities taking into consideration the remaining seats in the buses coming from the previous cities.

Example 1 (continued). If we assume that the capacity of each bus is 55 seats, and the number of students in each city is $s_1 = 42$, $s_2 = 81$, $s_3 = 87$, $s_4 = 437$ and $s_5 = 200$. Then, the optimal fleet of buses is $x_1^* = 1$, $x_2^* = 2$, $x_3^* = 2$, $x_4^* = 7$ and $x_5^* = 4$. But the optimal allocation of buses for coalition of cities $\{1, 4, 5\}$ is $x_1 = 1$, $x_4 = 8$ and $x_5 = 4$. Thus, if we consider a fixed cost of $F = 5000$ pesetas/bus (pts. of 1999), and a variable cost of $K = 143$ pts/km, then

$$c_{\mathcal{B}}(\overline{V}) = 16 \cdot 5000 + 1 \cdot 143k_1 + 2 \cdot 143k_2 + 2 \cdot 143k_3 + 7 \cdot 143k_4 + 4 \cdot 143k_5 \quad \text{and}$$
$$c_{\mathcal{B}}(\{1, 4, 5\}) = 13 \cdot 5000 + 1 \cdot 143k_1 + 8 \cdot 143k_4 + 4 \cdot 143k_5.$$

Now we can propose different mechanisms to distribute the total costs of each transport subsystem. It is sufficient to consider the cost sharing mechanism induced by the stand alone cost game together with some value of TU games as the solution concept for this class of games. This is the approach put forward by Shubik [27], and followed by Legros [17] and Sudhölter [28]. Once the costs have been allocated to the cities, the cost share of each city would be distributed equally among its students. However, we are aware of the impossibility of recovering all the information we need for explicitly defining the characteristic function of the stand alone cost game. Therefore we could only consider allocation rules which do not require the explicit knowledge of all coalition costs. In particular, we would propose sharing the total cost of the transport subsystem according to rules such as the *Egalitarian Nonseparable Cost (ENSC) method* or to the *Alternate Cost Avoided (ACA) method*, which are two separable cost allocation methods. For a discussion and interpretation of these rules, the reader may refer to [5,33]. More information on separable costs and related allocation methods can be found in [6,30].

Although there are conditions relating the ACA and ENSC methods with the τ-value [32] and the nucleolus [24] of the corresponding savings game (see [5]), it is easy to find examples to check that these games do not satisfy such conditions. In this case, we cannot easily analyze the properties of the ACA and ENSC methods. Moreover, this approach does not permit us to take into account all the special features of this problem. For instance, the agents have been involved in a strategic environment which is not suitable for the situation we deal with. We must remember that the organization responsible for setting the fees for using the system is the *Diputació*. Therefore we have decided to tackle the situation by means of an alternative approach. To be exact, we fall back on a cost sharing framework.

4. A cost sharing approach: aggregated egalitarian allocation rule

In the existing cost sharing literature, more important than the solutions themselves, are the principles of fairness which are represented there. This fact has led us to define a particular cost sharing rule which embraces the principles of fairness which every public institution should respect. The *Diputació* should avoid any resentment between students (and cities) arising from the proposed allocation rule. That is, such a rule should respect the basic statement: "all students are equal, and equals should be treated equally". The key to this basic statement is when two students may be considered as equal. For instance, if there were only fixed costs, then every student would use the system up to the same amount. Every student would use one bus. In such a case, all students would be equal. Since everybody should be treated equally, the total costs $F \cdot \sum_{i \in \overline{V}} x_i^*$ should be shared equally among the students. In order to extend this reasoning to a general tree buses situation, we will first distinguish two types of costs, fixed and variable costs, and then we will propose a method to allocate them among the users. We refer to this method as the *aggregated egalitarian solution.*

4.1. Variable costs allocation

First, we will focus on allocating the variable costs. If we wish to apply the principles of fairness described above to this problem, we should look for those partial subproblems in which all students can be considered as equals. We will thus allocate the total variable costs of the transport system by means of an aggregation process. First, we distribute the variable cost of each link – road joining two adjacent cities in the tree – among its users. Then, the variable cost share of each student will be obtained as the sum of all their partial shares.

Let $\mathcal{B} = (\mathrm{T}, \mathbf{s}, \mathbf{d}, b, H)$ be a tree buses situation. For every link $e = (i, j) \in E$, let N_e be the set of students whose bus pass through stage e. That is, the set of all students who live in any city located in "branch" $B_e = \{\ell \in \overline{V} \mid e \text{ is in path } P(\ell)\}$ of the tree routed at edge $e = (i, j) \in E$. Since every student of N_e is requesting the same service during link e, and everybody should be treated equally, the total variable costs related to this link

$$vc_e = \Big(\sum_{j \in B_e} x_j^*\Big) d_e K,$$

should be shared equally among the students in N_e. Thus, we define the *aggregated egalitarian allocation* of the variable costs as follows. Let $j \in \overline{V}$ be any city of the tree. Then, according to the egalitarian allocation, the cost shares of every student i living in city j will be given by

$$z_i^j = \sum_{e \in P_j} \frac{(\sum_{j \in B_e} x_j^*) d_e K}{n_e},$$

where $n_e = |N_e|$ is the number of students using link e, and P_j is defined as the set of links which are on the unique path in the tree T connecting city j to the root of the tree, the campus. The joint cost shares of all students living in j will be

$$z^j = s_j \Big(\sum_{e \in P_j} \frac{(\sum_{j \in B_e} x_j^*) d_e K}{n_e} \Big) \quad \text{for all } j \in \overline{V}.$$

Example 1 (continued). Let us consider the example of the *Marina Baixa* subsystem with distances $\mathbf{d} = (6, 9, 10, 9, 33)$ and $\mathbf{k} = (48, 61, 52, 42, 33)$. Then, taking into account that the variable cost is $K = 143$ pesetas per km and the optimal fleet is $x_1^* = 1,\ x_2^* = 2,\ x_3^* = 2,\ x_4^* = 7$ and $x_5^* = 4$, it holds that:

Link	Students	Total cost	Partial shares	City	Shares per est.
L'Alfàs–Benidorm	42	858	20.43	L'Alfàs	133.44
Calpe–Altea	81	2574	31.78	Calpe	178.84
Altea–Benidorm	168	5720	34.05	Altea	147.06
Benidorm–La Vila	647	15444	23.87	Benidorm	113.01
La Vila–Campus	847	75504	89.14	La Vila	89.14

Remark 1. The situation studied above generalizes a class of cost sharing problems which has been studied extensively, the problem of sharing the cost of a tree network. This problem has been studied within the context of the *standard tree games* (maintenance games). The study of these types of situations, or similar ones, has resulted in a long list of papers (see [2,4,9–12,16,20]). The special case when the underlying structure of the game is a chain, is also known as the airport problem and has been considered by several authors [1,18,19,29]. The generalization is given by the fact that in a standard

tree game, it is assumed that the maintenance cost of each link is the same, regardless of the number of users it serves.

The cost sharing method we have proposed has been deduced on the basis of a principle of fairness. Nevertheless, there are some questions which must be asked before assuming the fitness of the aggregated egalitarian solution. The most interesting properties in the cost sharing literature concern the dynamics of a cost sharing mechanism: how the cost sharing mechanism should adjust the individual cost shares if the parameters of the problem are perturbed. Obviously, the mechanism we have defined above, favours technological advances. It is trivial to check that the aggregated egalitarian rule verifies monotonicity in costs properties. Nevertheless, in this setting, the allocation rule should provide an incentive for agents to cooperate (population monotonicity properties). However, below we will give an illustrative example which shows some drawbacks of the allocation mechanism we have proposed.

Example 2. Let us consider the following situation, with buses of capacity 55 (figure 2).

Let us suppose that last year no student was living in city 2, therefore students living in cities 1 or 3 only needed 1 bus to reach the campus. This year 5 new students live in city 2. If they join the transport system, then the variable cost of link e is doubled. The partial variable cost share of each student living in city 1 or 3 is almost doubled, too. Hence, these 5 new students are “bad partners” for students in 1 or 3. However, since it is a public transport system, the five students of city 2 must be served. But the question is: who should pay the increase in costs, who should cover the originated deficit?

Example 2 shows a situation in which a group of students are subsidizing other students. This seems unfair. Students who are paying an additional subsidy to other students could feel that they are being treated unfairly. However, as we have pointed out before, the *Diputació* must provide the service to every student. Therefore it seems reasonable that this public institution should pay this additional subsidy. That is, we propose to create a compensatory monetary system.

4.1.1. Compensatory monetary system

If we define a cooperative cost game associated with the cost sharing problem we are analyzing (a variable cost tree game), and the aggregated egalitarian allocation turns out

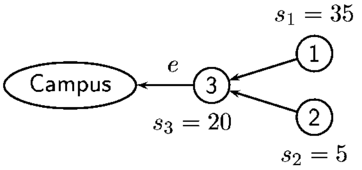

Figure 2. Example 2.

to be in the core of the game, then it is clear that in such a case no group of students would be subsidizing other students. Formally, the *core* of a cost game (N, c) is the set

$$C(N, c) = \{\mathbf{z} \in \mathbb{R}^N \mid z(S) \leqslant c(S) \text{ for all } S \subset N, \text{ and } z(N) = c(N)\},$$

where $z(S) = \sum_{i \in S} z_i$. If $\mathbf{z} \in C(N, c)$, then no coalition S has an incentive to split off if $\mathbf{z}$ is the proposed vector of cost shares. That is, a core allocation provides incentives for voluntary cooperation. However, each core condition can be rewritten as $z(S) \geqslant c(N) - c(N \setminus S)$, $S \subseteq N$. If this condition is violated for some S, then it could be said that coalition $N \setminus S$ is subsidizing S. Thus, even if there is no need to give the agents an incentive to cooperate, there is still an argument for a core on equity grounds.[1] The problem arises when the core of the corresponding cost game is empty. Then, to prevent unfair situations, some external financing is needed. We will look at a certain extension of the core for calculating the minimum financial support which is needed to avoid unfair situations like the example: the additive ε-tax core (introduced in [25,26]). The *additive ε-tax core* of a cost game (N, c) is the set

$$C_\varepsilon^+(c) = \{\mathbf{z} \in \mathbb{R}^N \mid z(S) \leqslant c(S) + \varepsilon|S| \text{ for all } S \subset N, \text{ and } z(N) = c(N)\}.$$

Since we have obtained the aggregated egalitarian allocation link by link, we will also calculate the financial support link by link. So let us define as many games, which we will refer to as *link games*, as links in the tree.

Definition 1. Let $\mathcal{B} = (\mathrm{T}, \mathbf{s}, \mathbf{d}, b, H)$ be a given tree bus situation. Then for each link $e = (i, j) \in E$, the *link game* (N_e, c_e) is defined as follows:

(i) The cost of any nonempty coalition $S \subsetneq N_e$ is given by the variable cost of the minimum number of buses needed to take all the students in S from city j to city i, i.e., $c_e(S) = \lceil s/b \rceil d_e K$, where s is the number of students in coalition S, and $\lceil s/b \rceil$ represents the upper integer part of s/b.

(ii) The cost of the grand coalition N_e will be exogeneously determined. It will be given by the variable cost related to link e (which is determined by the optimal fleet, obtained when solving problem (4) for $S = \overline{V}$). That is, $c_e(N_e) = (\sum_{j \in B_e} x_j^*) d_e K$.

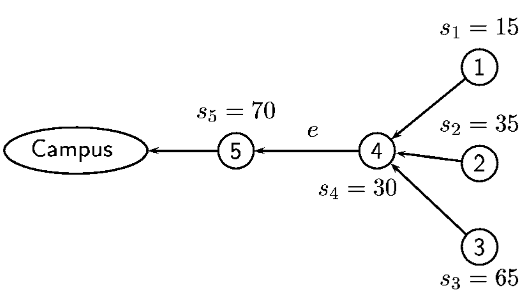

Figure 3. Example 3.

[1] This idea has been discussed extensively in the literature on public pricing. See references in [33].

Example 3. Let us consider the following situation with buses of capacity 55 (figure 3).

We only need $\lceil 145/55 \rceil = 3$ buses for moving all students in N_e from city 4 to city 5. In fact, however, there are $x_1^* + x_2^* + x_3^* + x_4^* = 1 + 1 + 2 + 0$ buses going through link e. Therefore the cost $c_e(N_e)$ must be equal to $4d_eK$, instead of $\lceil 145/55 \rceil d_eK$, in order to cover the real costs. This difference exists because there is a *surplus* $SB(e) = \sum_{j \in B_e} x_j^* - \lceil n_e/b \rceil$ *of 1 bus covering link* e. Note that the surplus of buses covering a link will be positive when, according to the optimal fleet $\mathbf{x}^*$, there were more buses covering link $e = (i, j)$ than the amount strictly needed to move all the students from city i to city j. The possible difference arises from the prohibition of transshipment in each node, because in the definition of the optimization problem (4) it is implicitly assumed that all contracted buses arrive at the university.

Remark 2. These games are similar to the type of games introduced and studied in [7]. There are two main differences between link games and "bus games". On the one hand, the costs of the grand coalition do not coincide. On the other hand, the agents of a "bus game" are travel agencies (cities), whereas the agents in a link game are students (travellers).

Theorem 1. Let $\mathcal{B} = (\mathrm{T}, \mathbf{s}, \mathbf{d}, b, H)$ be a tree buses situation and let $e \in E$. Then, the partial aggregated egalitarian allocation[2] is in the additive ε-tax core of the corresponding link game (N_e, c_e) if, and only if, $\varepsilon \geqslant \varepsilon_a$, with

$$\varepsilon_a = \begin{cases} 0 & \text{if } n_e \leqslant b \text{ and } SB(e) = 0, \\ \dfrac{SB(e)(n_e - 1) - 1}{n_e(n_e - 1)} d_eK & \text{if } n_e \leqslant b \text{ and } SB(e) \geqslant 1, \\ \dfrac{SB(e)}{n_e} d_eK & \text{if } n_e > b \text{ and } r_e = 0, \\ \left(\dfrac{SB(e)}{n_e} + \dfrac{b - r_e}{bn_e} \right) d_eK & \text{otherwise,} \end{cases}$$

where r_e is the remainder of the quotient n_e/b ($n_e = mb + r_e$, $m, r_e \in \mathbb{Z}_+$ and $0 \leqslant r_e < b$), and $SB(e) = \sum_{j \in B_e} x_j^* - \lceil n_e/b \rceil$ is the surplus of buses covering link e.

Proof. Let (N_e, c_e) be any link game. Then, taking $c_e(N_e) = (\sum_{j \in B_e} x_j^*) d_eK$ into account, it follows that the egalitarian allocation equals

$$z_i = \frac{(\sum_{j \in B_e} x_j^*) d_eK}{n_e} \quad \text{for all } i \in N_e.$$

On the other hand, the cost of coalition $S \subseteq N_e$ is given by $c_e(S) = \lceil s/b \rceil d_eK$, where s is the number of students in S; whereas the stand alone cost is $c_e(\{i\}) = d_eK$, for every

[2] Here, by the partial aggregated egalitarian allocation we refer to the cost shares related to link e prescribed by the aggregated egalitarian allocation. In the sequel we will refer to it as "egalitarian allocation".

student $i \in N_e$. Therefore, the egalitarian allocation is in the additive ε-tax core if, and only if,

$$\frac{(\sum_{j\in B_e} x_j^*)d_eK}{n_e}\cdot s \leqslant \left\lceil \frac{s}{b}\right\rceil d_eK + \varepsilon\cdot s \quad \text{for all } S \subsetneq N_e.$$

That is, for all $s \in \{1,\dots,n_e-1\}$, must be

$$\varepsilon \geqslant \frac{(\sum_{j\in B_e} x_j^*/n_e)s - \lceil s/b\rceil}{s} d_eK = \frac{(\sum_{j\in B_e} x_j^*)\, s - \lceil s/b\rceil\, n_e}{sn_e} d_eK.$$

Equivalently, taking $\sum_{j\in B_e} x_j^* = SB(e) + \lceil n_e/b\rceil$, into consideration

$$\varepsilon \geqslant \frac{SB(e)}{n_e} d_eK + \frac{\lceil n_e/b\rceil s - \lceil s/b\rceil n_e}{sn_e} d_eK. \qquad (\mathrm{Ia}_S)$$

Let us show that ε_a is the minimum ε satisfying (Ia_S) for all $S \subseteq N_e$. In fact, since the first right-hand term does not depend on S, it suffices to give an upper bound, $\tilde{\varepsilon}$, for the second one, i.e.,

$$\tilde{\varepsilon} \geqslant \frac{\lceil n_e/b\rceil s - \lceil s/b\rceil n_e}{sn_e} d_eK = g(s) \quad \text{for all } s \in \{1,\dots,n_e-1\}. \qquad (\mathrm{I'a}_S)$$

Three cases can arise:

(A) If $n_e \leqslant b$, then inequalities ($\mathrm{I'a}_S$) can be rewritten as follows

$$\tilde{\varepsilon} \geqslant \frac{s-n_e}{sn_e} d_eK = g(s).$$

Since $g(\cdot)$ is an increasing function, then

$$\tilde{\varepsilon} \geqslant g(n_e-1) = -\frac{1}{(n_e-1)n_e} d_eK.$$

Therefore, an upper bound ε for the original sum must satisfy

$$\varepsilon \geqslant \frac{SB(e)}{n_e} d_eK - \frac{1}{(n_e-1)n_e} d_eK = \frac{SB(e)(n_e-1)-1}{(n_e-1)n_e} d_eK.$$

Now, if $SB(e) = 0$, then $g(s) < 0$ and inequalities (Ia_S) are fulfilled for all $\varepsilon \geqslant 0 = \varepsilon_a$. Otherwise,

$$\varepsilon_a = \frac{(SB(e)(n_e-1)-1)d_eK}{(n_e-1)n_e}.$$

(B) If $n_e > b$ and n_e is a multiple of b ($m > 1$ and $r_e = 0$), then inequalities ($\mathrm{I'a}_S$) can be rewritten as follows

$$\tilde{\varepsilon} \geqslant \frac{ms - \lceil s/b\rceil mb}{sn_e} d_eK.$$

Let α and δ be two nonnegative integers such that $s = \alpha b + \delta$ and $\delta < b$. Two cases are possible:

(B.1) if $\alpha > 0$ and $\delta = 0$, then $g(s) = 0$;

(B.2) if $\alpha \geqslant 0$ and $\delta > 0$, then $g(s) < 0$ follows from being $\delta < b$.

Therefore, the inequalities (I′a$_S$) hold for all $\tilde{\varepsilon} \geqslant 0$ and, consequently, $\varepsilon_a = (SB(e)/n_e)\, d_e K$.

(C) Otherwise, $n_e = mb + r_e$ with $1 \leqslant r_e \leqslant b-1$ and $m \geqslant 1$, inequalities (I′a$_S$) can be rewritten as follows

$$\tilde{\varepsilon} \geqslant \frac{(m+1)s - \lceil s/b \rceil (mb + r_e)}{s n_e}\, d_e K \quad \text{for all } s \in \{2, \dots, n_e - 1\}.$$

Let α and δ be two nonnegative integers such that $s = \alpha b + \delta$ and $\delta < b$. As in the previous case, two possibilities can be distinguished:

(C.1) If $\alpha > 0$ and $\delta = 0$, then

$$\varepsilon \geqslant \frac{SB(e)}{n_e} d_e K + \tilde{\varepsilon} \geqslant \frac{SB(e)}{n_e} d_e K + \frac{b - r_e}{b n_e} d_e K = \varepsilon_a \quad \text{for all } s \text{ with } s = \alpha b. \tag{5}$$

(C.2) If $\alpha \geqslant 0$ and $\delta > 0$, then

$$\tilde{\varepsilon} \geqslant \frac{\delta(m+1) + \alpha(b - r_e) - n_e}{(\alpha b + \delta) n_e} d_e K = G_a(\alpha, \delta).$$

$G_a(\alpha, \cdot) := g^a_\alpha(\cdot)$ is an increasing function for every fixed $\alpha \geqslant 0$. Therefore, it holds

$$G_a(\alpha, \delta) \leqslant G_a(\alpha, b) \quad \text{for all } \alpha \geqslant 0 \text{ and } \delta \in \{1, \dots, b-1\}.$$

It can be easily checked that $G_a(\alpha, b) = ((b - r_e)/b n_e) d_e K$, for all $\alpha \geqslant 0$. Thus,

$$\begin{aligned} \varepsilon_a &= \frac{SB(e)}{n_e} d_e K + \frac{b - r_e}{b n_e} d_e K \geqslant \frac{SB(e)}{n_e} d_e K + G_a(\alpha, \delta) \\ &= \frac{SB(e)}{n_e} d_e K + \frac{\lceil n_e/b \rceil s - \lceil s/b \rceil n_e}{s n_e} d_e K, \end{aligned} \tag{6}$$

for all $\delta \in \{1, \dots, b-1\}$ and for all $\alpha \in \{0, \dots, b-1\}$.

Then, it follows from (5) and (6),

$$\varepsilon_a = \frac{SB(e)}{n_e} d_e K + \frac{b - r_e}{b n_e} d_e K = \left(\frac{SB(e)}{n_e} + \frac{b - r_e}{b n_e} \right) d_e K. \qquad \square$$

The previous theorem shows that the financial support that each student receives depends on the number of vacant seats per bus covering the link, as well as the number of spare buses (according to the optimal fleet). For instance, if we go back to examples 2 and 3, and we calculate the financial support per link, then: with respect to example 2, link (2,3) is not financed, whereas the financial support of link $e = (3, \text{Campus})$ equals the price of the $55 - 5$ extra seats of the bus departing from city 2; with respect to example 3, the financial support of link e equals the cost of the extra bus plus the cost of the 20 seats which would be vacant even if only 3 buses were used.

Example 1 (continued). The *Diputació* financing would be given by:

Link	L'Alfàs–Ben.	Calpe–Altea	Altea–Ben.	Ben.–La Vila	La Vila–Campus
Fin./student	0	8.38	8.05	0.47	3.34
Fin./link	0.00	678.6	1352	304.2	2831.4

Note that the total financial support (5166.2 pts.) represents 5.16% of the total variable cost of the transport subsystem (100100 pts.). This amount will be covered by the *Diputació d'Alacant*. Then the *financed variable cost share* that each student should pay is:

L'Alfàs del Pi	Calpe	Altea	Benidorm	La Vila Joiosa
129.63	158.6	135.2	109.2	85.8

The next proposition shows the amount that each student should pay per kilometer, and per link, once the *Diputació* has financed them.

Proposition 1. Let $\mathcal{B} = (\mathrm{T}, \mathbf{s}, \mathbf{d}, b, H)$ be a tree buses situation and let $e \in E$. Then the fare per kilometer that each student should pay after subtracting financial support is given by

$$\beta_e = \begin{cases} \dfrac{1}{n_e} K & \text{if } n_e \leqslant b \text{ and } SB(e) = 0, \\ \dfrac{1}{n_e - 1} K & \text{if } n_e \leqslant b \text{ and } SB(e) \geqslant 1, \\ \dfrac{1}{b} K & \text{if } n_e > b. \end{cases}$$

Proof. The variable cost per kilometer that each student should pay after subtracting financial support is given by

$$\frac{((\sum_{j \in B_e} x_j^*)/n_e) \cdot d_e K - \varepsilon_a}{d_e}.$$

Then, taking this amount into account the statement can be easily established. □

4.2. Fixed costs allocation

At the beginning of this section we have argued that if there were only fixed costs, then the total costs $F \cdot \sum_{i \in \overline{V}} x_i^*$ should be shared equally among the students. Therefore, since the variable cost have already been allocated, an egalitarian allocation of fixed costs should prescribe to divide the total costs equally. However, the same problems arise in this case. It could be the case that some students were subsidizing other students. So, to avoid such an unfair situation, we will consider an *associated fixed cost game*, (N, c_f), defined as follows.

Let $\mathcal{B} = (\mathrm{T}, \mathbf{s}, \mathbf{d}, b, H)$ be a tree buses situation, and let $N = \{1, \ldots, n\}$ be the set of all students living in the cities located in the tree. Then, the total fixed costs $c_f(N) := (\sum_{j\in\overline{V}} x_j^*) \cdot F$ have to be distributed among students in N. However, each coalition of students $S \subsetneq N$ with $|S| = s$ only needs $\lceil s/b \rceil$ buses[3]. Thus, the maximum amount that students in S are willing to pay is given by $c_f(S) := \lceil s/b \rceil \cdot F$.

Note that the associated fixed costs game is equal to a link game (N_e, c_e) with $N_e = N$ and $d_e = F$. Therefore, all previous results can be applied.

According to theorem 1, the amount per student financed by the *Diputació* has to be given by

$$\varepsilon_f = \begin{cases} 0 & \text{if } n \leqslant b \text{ and } SB = 0, \\ \dfrac{SB(n-1)-1}{n(n-1)} F & \text{if } n \leqslant b \text{ and } SB \geqslant 1, \\ \dfrac{SB}{n} F & \text{if } n > b \text{ and } n \text{ is a multiple of } b, \\ \left(\dfrac{SB}{n} + \dfrac{b-r}{bn}\right) F & \text{otherwise,} \end{cases}$$

where r is the remainder of the quotient n/b and $SB = (\sum_{j\in\overline{V}} x_j^*) - \lceil n/b \rceil$ is the *surplus of buses covering the routes*, which can be interpreted as follows.

If the variable cost to be paid per kilometer covered was equal to zero, then the structure of the tree would not influence in the determination of the optimal fleet. Specifically, the costs would only depend on the number of buses hired. Therefore, the optimal strategy would be to contract $\lceil n/b \rceil$. Then, the surplus of buses covering the routes can be read as the surplus of buses which have to be hired for reducing the variable costs (as well as the time spent by the students on the trip).

Thus, the *financed fixed costs share* of each student will be given by

$$\beta = \begin{cases} \dfrac{F}{n} & \text{if } n \leqslant b \text{ and } SB = 0, \\ \dfrac{F}{n-1} & \text{if } n \leqslant b \text{ and } SB \geqslant 1, \\ \dfrac{F}{b} & \text{if } n > b. \end{cases}$$

[3] Note that since we are not dealing with variable costs, the distance covered by a bus is not being considered.

Example 1 (continued). If we consider a fixed cost of 5000 pesetas per bus, then each student must pay $5000/55 = 90.91$ pesetas, whereas the financial support given by the *Diputació* has to be $33 \cdot (5000/55)$ (the 33 seats which are free in a fleet of 16 buses, with 55 seats, used by 847 students). Therefore, the total financial support (3000 pts.) represents 3.75% of the total fixed costs of the transport system ($16 \cdot 5000 = 80000$ pts.).

4.3. *Financed aggregated egalitarian rule*

Up to the present time, we have determined the total financial support that the *Diputació* should accept in order to transform the aggregated egalitarian allocation into a stable one: the amount it has to finance of the variable costs of every link, as well as the amount corresponding to the fixed costs. Formally, let $j \in \overline{V}$ be a city of the tree. Then, the *financed cost shares* of every student i living in city j will be given by

$$y_i^j = \beta + \sum_{e \in P_j} \beta_e d_e,$$

whereas the joint (financed) cost shares of all students living in j will be

$$y^j = s_j \left(\beta + \sum_{e \in P_j} \beta_e d_e \right) \quad \text{for all } j \in \overline{V}. \tag{7}$$

Finally, the total financial support required will be

$$FS(\mathcal{B}) = n\varepsilon_f + \sum_{e \in E} n_e \varepsilon_a(e),$$

where $\varepsilon_a(e)$ is the financial support per student over link e (see theorem 1).

The partial results we have obtained previously can be summarized in the following theorem. First, we define a *total financed tree buses game*. Second, we prove that the proposed fare per city, see (7), turns out to be a core allocation of the financed game.

Definition 2. Let $\mathcal{B} = (\mathrm{T}, \mathbf{s}, \mathbf{d}, b, H)$ be a tree buses situation and let $(\overline{V}, c_{\mathcal{B}})$ be the corresponding tree buses game played by the cities (see p. 5). Then, we define the *financed tree buses game*, $(\overline{V}, c_{\mathcal{B}}^f)$ as follows:

$$c_{\mathcal{B}}^f(S) := c_{\mathcal{B}}(S) \quad \text{for all } S \subsetneq \overline{V},$$
$$c_{\mathcal{B}}^f(\overline{V}) := c_{\mathcal{B}}(\overline{V}) - FS(\mathcal{B}).$$

Note that students receive financial support from the *Diputació* if, and only if, they all cooperate.

Theorem 2. Let $\mathcal{B} = (\mathrm{T}, \mathbf{s}, \mathbf{d}, b, H)$ be a tree buses situation. Then, the allocation $\mathbf{y} = (y^j)_{j \in \overline{V}}$ given by expression (7) belongs to the core of the financed tree buses game $(\overline{V}, c_{\mathcal{B}}^f)$.

Proof. Since the total financial support $FS(\mathcal{B})$ is obtained as the sum of every student's support, $\mathbf{y}$ is efficient. Let us show it. On the one hand,

$$y(\overline{V}) := \sum_{j\in\overline{V}} y_j = \sum_{j\in\overline{V}} s_j\Big(\beta + \sum_{e\in P_j}\beta_e d_e\Big) = n\beta + \sum_{j\in\overline{V}} s_j\Big(\sum_{e\in P_j}\beta_e d_e\Big).$$

Then, since a link e is in the path of j, P_j, if, and only if, j is in the branch B_e, it follows

$$y(\overline{V}) = n\beta + \sum_{e\in E}\Big(\sum_{j\in B_e} s_j\Big)\beta_e d_e = n\beta + \sum_{e\in E} n_e\beta_e d_e. \tag{8}$$

On the other hand,

$$c^f_{\mathcal{B}}(\overline{V}) := c^c_{\mathcal{B}}(\overline{V}) - FS(\mathcal{B}) = F\sum_{j\in\overline{V}} x^*_j + K\sum_{j\in\overline{V}} x^*_j k_j - \Big(n\cdot\varepsilon_f + \sum_{e\in E} n_e\varepsilon_a(e)\Big).$$

Taking $SB = \sum_{j\in\overline{V}} x^*_j - \lceil n/b\rceil$ and $k_j = \sum_{e\in P_j} d_e$ into account, it follows

$$c^f_{\mathcal{B}}(\overline{V}) = \Big(\Big\lceil\frac{n}{b}\Big\rceil + SB\Big)F - n\varepsilon_f + \sum_{e\in E}\Big(d_e K\sum_{j\in B_e} x^*_j - n_e\varepsilon_a(e)\Big),$$

which coincides with

$$\Big(\Big\lceil\frac{n}{b}\Big\rceil + SB\Big)F - n\varepsilon_f + \sum_{e\in E}\Big(d_e K\Big(\Big\lceil\frac{n_e}{b}\Big\rceil + SB(e)\Big) - n_e\varepsilon_a(e)\Big) \tag{9}$$

because $SB(e) = \sum_{j\in B_e} x^*_j - \lceil n_e/b\rceil$. Therefore, it follows from (8), (9) and

(i) $n\beta = (\lceil n/b\rceil + SB)F - n\varepsilon_f$,

(ii) $n_e\beta_e d_e = d_e K(\lceil n_e/b\rceil + SB(e)) - n_e\varepsilon_a(e)$ for all $e \in E$,

that $y(\overline{V}) = c^f_{\mathcal{B}}(\overline{V})$. Now, we prove that $\mathbf{y}$ is stable. Let $S \subsetneq N$ be any nonempty coalition of players. We denote by $E(S) = \{e \in E \mid S \cap B_e \neq \emptyset\}$ the set of all links which students in S have to cover to reach the campus. Then,

$$y(S) := \sum_{j\in S} y_j = \sum_{j\in S} s_j\beta + \sum_{j\in S} s_j\Big(\sum_{e\in P_j}\beta_e d_e\Big) = \beta\sum_{j\in S} s_j + \sum_{e\in E(S)} d_e\beta_e\Big(\sum_{j\in B_e\cap S} s_j\Big).$$

The cost of coalition S is given by

$$\begin{aligned}
c^f_{\mathcal{B}}(S) := c_{\mathcal{B}}(S) &= F\sum_{j\in S} x^*(S) + K\sum_{j\in S} x^*(S)k_j \\
&= F\sum_{j\in S} x^*(S) + K\sum_{j\in S} x^*(S)\Big(\sum_{e\in P_j} d_e\Big) \\
&= F\sum_{j\in S} x^*(S) + K\sum_{e\in E(S)} d_e\Big(\sum_{j\in B_e\cap S} x^*(S)\Big),
\end{aligned}$$

where $\mathbf{x}^*(S)$ is the optimal allocation of buses which determines the cost $c_{\mathcal{B}}(S)$. Therefore, the inequality $y(S) \leqslant c^f_{\mathcal{B}}(S)$ is obtained from the following inequalities:

(i) $\beta \sum_{j \in S} s_j \leqslant F \sum_{j \in S} x^*(S)$,

(ii) $d_e \beta_e (\sum_{j \in B_e \cap S} s_j) \leqslant d_e K (\sum_{j \in B_e \cap S} x^*(S))$ for all $e \in E(S)$. □

Example 1 (continued). The subsidised fare that each student should pay is:

L'Alfàs del Pi	Calpe	Altea	Benidorm	La Vila Joiosa
220.54	249.51	226.11	200.11	176.71

and the fare that each student should pay without subsidy is:

L'Alfàs del Pi	Calpe	Altea	Benidorm	La Vila Joiosa
227.89	273.29	241.52	207.46	183.59

These figures have been calculated according to the prices which we have been able to obtain when hiring (privately) only 1 bus. Obviously, if the project we have proposed to the *Diputació* (UNIVERCITY Project) is developed, the costs would be drastically reduced. Nevertheless, even in this situation (with the high prices), the fare that each student paid in 1999 for a similar service (see table below) is greater than the fare prescribed by the aggregated egalitarian solution, with and without subsidy, for all students except Benidorm students. Since Benidorm students contracted 8 buses (instead of 1), they were able to obtain lower prices.

L'Alfàs del Pi	Calpe	Altea	Benidorm	La Vila Joiosa
240	295	245	195	187.5

5. Concluding remarks

The transport system for university students in the province of *Alacant* introduced in this paper is called An Integrated Transport System (UNIVERCITY) since it consists of a joint transport system among town councils in the *Alacant* province, managed by the *Diputació*, which tries to connect villages and towns in *Alacant* efficiently with the different university campuses.

When computing the financial support to be covered by the *Diputació*, we could have followed another approach. To be exact, we could have considered the multiplicative ε-tax core (introduced by Tijs and Driessen [31]). In fact, we have also considered this approach in previous versions of this paper. Nevertheless, the financial support, as well as the financed cost sharing rule which we have obtained coincide. The reader interested in the equivalent multiplicative approach may refer to [23].

It should be noted that the model we have proposed in this paper is a first approach to the real problem. The whole project has been presented to the *Diputació d'Alacant* and is being evaluated. If the project were to be developed, it would permit collecting all the necessary data. This would therefore allow us to study possible generalizations of the model described here. For instance, a possible extension would concern the timetables. We have considered that each student goes on a return-trip except during holidays. The real meaning behind this assumption should be considered carefully. In fact, there is no problem since each student must pay a monthly fare, no matter the days they take the bus. Even so, at least two timetables should be taken into consideration, one in the morning and one in the afternoon. How does this situation modify the allocation? A new situation could be studied in which each tree appears twice but with different information (morning and afternoon trees), or another in which different trees are used, and so forth. Another possible extension could be given by considering more complicated transport subsystems. We could consider subsystems with two Campuses. One of them would be the root of the tree, whereas the other one would be the terminus of some routes and, at the same time, the intermediate bus stop of some other routes.

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