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Estimating GARCH volatility in the presence of outliers

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ABSTRACT

GARCH volatilities depend on the unconditional variance, which is a non-linear function of the parameters. Consequently, they can have larger biases than estimated parameters. Using robust methods to estimate both parameters and volatilities is shown to outperform Maximum Likelihood procedures.

1. Introduction

When GARCH models are fitted to real data, the residuals often have excess kurtosis which could be explained, among other reasons, by the presence of outliers; see Baillie and Bollerslev (1989). Many authors argue that extreme observations are not outliers and they should be incorporated into the model; see Eraker et al. (2003). However, since in practice, GARCH models are often fitted without taking into account whether there are observations generated by a different stochastic mechanism, we find it interesting to analyze how such observations affect the estimation of the volatility.

To deal with outliers, one could identify and correct them before estimating the GARCH parameters; see Grané and Veiga (2010) and Gregory and Reeves (2010). However, all proposed detection methods suffer from a number of potential problems. Another alternative is using robust methods. Maximizing the likelihood based on a heavy tailed distribution protects against outliers when estimating the GARCH parameters; see Sakata and White (1998), Karanasos and Kim (2006) and Carnero et al. (2007). Bollerslev (1987) proposed a Quasi Maximum Likelihood estimator, denoted by QML-t, based on maximizing the Student-t log-likelihood function. Alternatively, Muler and Yohai (2008) proposed the Bounded-M (BM) estimator, which is defined by the maximization of a conveniently modified likelihood and changes the specification of the conditional variance to bound the propagation of the outlier effect.

Previous research has focused on the effects of outliers on the Gaussian Maximum Likelihood (ML) estimator of GARCH parameters; see Sakata and White (1998), Carnero et al. (2007) and Muler and Yohai (2008). However, the main interest of practitioners is the estimation of the underlying volatilities. Our objective is to study the effects of outliers on the estimated GARCH volatilities computed by using ML estimates of the parameters and compare the performance of robust estimators.

2. Effects of outliers on ML volatility estimates

Consider a GARCH(1, 1) series contaminated by k consecutive outliers of size ω at times $\tau, \ldots, \tau + k - 1$:

$$y_t = y_t^* + \mathrm{sign}(y_t^*)\omega I_t \tag{1}$$

with

$$y_t^* = \varepsilon_t \sigma_t$$

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Table 1
Monte Carlo means and standard deviations of GARCH estimates when $\alpha_0 = 0.1$, $\alpha_1 = 0.1$, $\beta = 0.8$ and the outlier size is $\omega = 10$.

		A single outlier				Two consecutive outliers			
		ML	QML-*t*	BQML-*t*	BM	ML	QML-*t*	BQML-*t*	BM
$T = 500$	α_0	0.4193 (0.3509)	0.3657 (0.2971)	0.1907 (0.1962)	0.2631 (0.2763)	0.5930 (0.3823)	0.5138 (0.2848)	0.5063 (0.2999)	0.2832 (0.2975)
	α_1	0.1480 (0.1396)	0.1047 (0.0538)	0.1216 (0.0536)	0.1125 (0.0542)	0.2159 (0.1157)	0.1730 (0.0545)	0.1695 (0.0546)	0.1060 (0.0555)
	β	0.5196 (0.3157)	0.6523 (0.2624)	0.7143 (0.2014)	0.6151 (0.2906)	0.3309 (0.3060)	0.3716 (0.2653)	0.3788 (0.2866)	0.5998 (0.3095)
$T = 1000$	α_0	0.2528 (0.2412)	0.2164 (0.1847)	0.1317 (0.0889)	0.1581 (0.1682)	0.3499 (0.2670)	0.2919 (0.1850)	0.2874 (0.1896)	0.1697 (0.1833)
	α_1	0.1163 (0.0692)	0.1070 (0.0386)	0.1148 (0.0354)	0.1075 (0.0356)	0.1610 (0.0611)	0.1436 (0.0348)	0.1427 (0.0359)	0.1070 (0.0348)
	β	0.6600 (0.2378)	0.6854 (0.1868)	0.7680 (0.1031)	0.7321 (0.1840)	0.5331 (0.2590)	0.5804 (0.1905)	0.5859 (0.2010)	0.7196 (0.2006)
$T = 5000$	α_0	0.1164 (0.0308)	0.1116 (0.0230)	0.1056 (0.0218)	0.1033 (0.0226)	0.1276 (0.0346)	0.1253 (0.0438)	0.1238 (0.0250)	0.1029 (0.0222)
	α_1	0.1019 (0.0173)	0.1016 (0.0136)	0.1090 (0.0143)	0.1029 (0.0144)	0.1095 (0.0169)	0.1090 (0.0135)	0.1094 (0.0129)	0.1058 (0.0147)
	β	0.7851 (0.0405)	0.7886 (0.0316)	0.7916 (0.0301)	0.7949 (0.0316)	0.7673 (0.0433)	0.7683 (0.0415)	0.7694 (0.0324)	0.7944 (0.0316)

where ε_t is a Gaussian white noise, I_t takes value 1 when $t = \tau, \ldots, \tau + k - 1$ and 0 otherwise, and

$$\sigma_t^2 = \alpha_0 + \alpha_1 y_{t-1}^{*2} + \beta \sigma_{t-1}^2 \tag{2}$$

with $\alpha_0 > 0$, $\alpha_1 \geq 0$, $\beta \geq 0$ and $\alpha_1 + \beta < 1$.[1]

Denote by $\left\{\widehat{\alpha}_0^{\mathrm{ML}}, \widehat{\alpha}_1^{\mathrm{ML}}, \widehat{\beta}^{\mathrm{ML}}\right\}$ the ML estimator of the parameters. ML estimated volatilities are given by

$$\left(\widehat{\sigma}_t^{\mathrm{ML}}\right)^2 = \widehat{\alpha}_0^{\mathrm{ML}} + \widehat{\alpha}_1^{\mathrm{ML}} y_{t-1}^2 + \widehat{\beta}^{\mathrm{ML}} \left(\widehat{\sigma}_{t-1}^{\mathrm{ML}}\right)^2. \tag{3}$$

Consider an isolated outlier at $t = \tau$. The error in the estimation of σ_t^2 is given by

$$\begin{aligned}
\xi_t &= \left(\widehat{\sigma}_t^{\mathrm{ML}}\right)^2 - \sigma_t^2 \\
&= (\widehat{\alpha}_0^{\mathrm{ML}} - \alpha_0) + (\widehat{\alpha}_1^{\mathrm{ML}} - \alpha_1) y_{t-1}^2 \\
&\quad + (\widehat{\beta}^{\mathrm{ML}} - \beta)\sigma_{t-1}^2 + \widehat{\beta}^{\mathrm{ML}} \left(\left(\widehat{\sigma}_{t-1}^{\mathrm{ML}}\right)^2 - \sigma_{t-1}^2\right) \\
&= \frac{(\widehat{\alpha}_0^{\mathrm{ML}} - \alpha_0)\left(1 - \left(\widehat{\beta}^{\mathrm{ML}}\right)^{t-1}\right)}{1 - \widehat{\beta}^{\mathrm{ML}}} \\
&\quad + (\widehat{\alpha}_1^{\mathrm{ML}} - \alpha_1) \sum_{i=0}^{t-2} \left(\widehat{\beta}^{\mathrm{ML}}\right)^i y_{t-1-i}^{*2} \\
&\quad + (\widehat{\beta}^{\mathrm{ML}} - \beta) \sum_{i=0}^{t-2} \left(\widehat{\beta}^{\mathrm{ML}}\right)^i \sigma_{t-1-i}^2 \\
&\quad + \widehat{\alpha}_1^{\mathrm{ML}} \sum_{i=0}^{t-2} \left(\widehat{\beta}^{\mathrm{ML}}\right)^i (y_{t-1-i}^2 - y_{t-1-i}^{*2}) \\
&\quad + \left(\widehat{\beta}^{\mathrm{ML}}\right)^{t-1} \left(\left(\widehat{\sigma}_1^{\mathrm{ML}}\right)^2 - \sigma_1^2\right).
\end{aligned} \tag{4}$$

Note that the expected error in the volatility estimation depends on the biases, covariances and expectations of non-linear functions of the ML estimator. Although the biases of the ML estimator due to outliers have been already analyzed, for completeness, we have carried out a Monte Carlo experiment generating 1000 series, of sizes $T = 500$, 1000 and 5000, by a GARCH(1, 1) model with parameters $\alpha_0 = 0.1$, $\alpha_1 = 0.1$ and $\beta = 0.8$. The series have been contaminated at $\tau = T/2$ by first an isolated and second by two consecutive outliers of sizes $\omega = 0$, 5, 10 and 15. As the marginal variance of the uncontaminated series is 1, ω represents the size of the outlier in terms of the number of standard deviations. Table 1, which reports the Monte Carlo means and standard deviations of the ML estimates when $\omega = 10$, shows that, if the sample size is $T = 1000$, the biases of $\widehat{\alpha}_0^{\mathrm{ML}}$ and $\widehat{\alpha}_1^{\mathrm{ML}}$ are positive and large while $\widehat{\beta}^{\mathrm{ML}}$ has a very large negative bias. The biases are larger when the outliers are consecutive. Furthermore, Fig. 1, which plots kernel densities of $\widehat{\alpha}_0^{\mathrm{ML}}$, $\widehat{\alpha}_1^{\mathrm{ML}}$ and $\widehat{\beta}^{\mathrm{ML}}$ and of the estimated marginal variance given by $\widehat{\alpha}_0^{\mathrm{ML}}/(1 - \widehat{\alpha}_1^{\mathrm{ML}} - \widehat{\beta}^{\mathrm{ML}})$ for $T = 1000$ when there is an isolated outlier, shows that when $\omega = 15$, both $\widehat{\alpha}_1^{\mathrm{ML}}$ and $\widehat{\beta}^{\mathrm{ML}}$ can take any value within the admissible parameter space. Notice also the large bias of the estimated marginal variance, its density is concentrated so far from the true value that only the left tail is plotted. Consequently, if $\left(\widehat{\sigma}_1^{\mathrm{ML}}\right)^2$ is the estimated marginal variance, the volatilities will be positively biased as illustrated in Fig. 2 where the biases of the Monte Carlo estimated volatilities are plotted for both single and consecutive outliers. At $t = \tau + 1$, the error is positive because y_τ^2 is large while σ_τ^2 is not affected by the outlier; see (4). Therefore, we expect that the volatility is overestimated by a very large amount right after the outlier appears. Then, at $t > \tau + 1$, the error is increased due to the term $\widehat{\alpha}_1^{\mathrm{ML}} \sum_{i=0}^{t-2} \left(\widehat{\beta}^{\mathrm{ML}}\right)^i (y_{t-1-i}^2 - y_{t-1-i}^{*2})$ that takes into account the difference between the contaminated observations used to estimate the variances and the uncontaminated observations that enter the equation of the true variances. Given that $y_\tau^2 - y_\tau^{*2} > 0$, there is transmission of the effects of outliers at time τ to estimated volatilities after $\tau + 1$. Then, the errors tend exponentially towards their previous mean; see Fig. 2. Note that, if $\omega = 15$, ML estimated volatility overestimates the true volatility by more than 25% at any time t and by more than 100% right after the outlier. The pattern is similar regardless of whether the outliers are single or consecutive.

3. Robust estimation of the volatility

Muler and Yohai (2008) propose the following volatility estimator

$$\begin{aligned}
\left(\widehat{\sigma}_t^{\mathrm{BM}}\right)^2 &= \widehat{\alpha}_0^{\mathrm{BM}} + \widehat{\alpha}_1^{\mathrm{BM}} r_c\left(\frac{y_{t-1}^2}{\left(\widehat{\sigma}_{t-1}^{\mathrm{BM}}\right)^2}\right) \\
&\quad \times \left(\widehat{\sigma}_{t-1}^{\mathrm{BM}}\right)^2 + \widehat{\beta}^{\mathrm{BM}} \left(\widehat{\sigma}_{t-1}^{\mathrm{BM}}\right)^2
\end{aligned} \tag{5}$$

where $r_c(x) = \begin{cases} x, & |x| < c \\ c, & |x| \geq c \end{cases}$ and $\widehat{\alpha}_0^{\mathrm{BM}}$, $\widehat{\alpha}_1^{\mathrm{BM}}$ and $\widehat{\beta}^{\mathrm{BM}}$ are given by the BM estimator.[2] Table 1, which reports their Monte Carlo means and standard deviations for the same experiments described above, shows that the biases, which are similar regardless of whether the outliers are single or consecutive, are clearly reduced with respect to ML; see also Fig. 1.

Alternatively, we propose to estimate the volatility in (5) with

$$r_c(x) = \begin{cases} x, & |x| < c \\ 1, & |x| \geq c \end{cases} \tag{6}$$

[1] This specification corresponds to Level Outlier defined by Hotta and Tsay (1998) who also define Volatility Outliers.

[2] Muler and Yohai (2008) show that the BM estimator outperforms alternative robust estimators proposed by Park (2002) and Peng and Yao (2003).

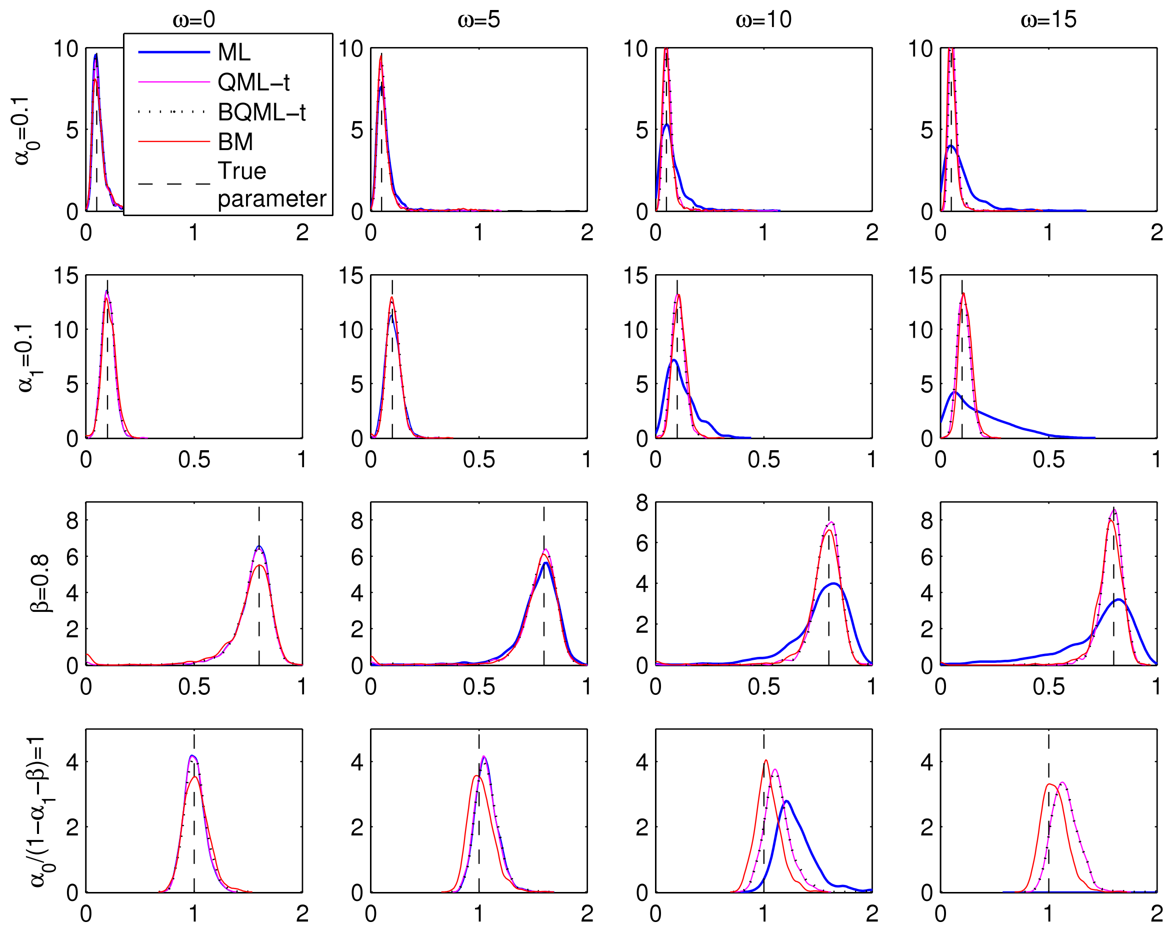

Fig. 1. Monte Carlo kernel densities of estimated parameters of simulated GARCH(1, 1) series of size $T = 1000$ contaminated with one outlier.

and the parameters being estimated by the QML-*t* estimator. Using (6), large squared observations are set equal to their conditional expectations, which are given by the conditional variance. We consider a third robust estimator of the parameters, denoted by BQML-*t*, based on combining the maximization of the Student-*t* log-likelihood with the bounding mechanism proposed by Muler and Yohai (2008). The resulting volatility estimates are denoted by $\left(\widehat{\sigma}_t^{\text{QML-}t}\right)^2$ and $\left(\widehat{\sigma}_t^{\text{BQML-}t}\right)^2$ respectively. Table 1 shows that the QML-*t* estimator has smaller bias than ML. However, the biases of $\widehat{\alpha}_0^{\text{QML-}t}$ and $\widehat{\beta}^{\text{QML-}t}$ are still too large. The results corresponding to the BQML-*t* estimator with $c = 9$, show that the biases of $\widehat{\alpha}_0^{\text{BQML-}t}$ and $\widehat{\beta}^{\text{BQML-}t}$ are further reduced when there is a single outlier. However, when there are two consecutive outliers, the biases of the QML-*t* and BQML-*t* estimators are very similar. Fig. 1 shows that when $\omega = 0$, the densities of the ML and the robust estimators are similar. Moreover, in the presence of outliers, $\widehat{\beta}^{\text{QML-}t}$ has a dispersed density which is reduced by the bounding mechanism introduced by the BQML-*t* and BM estimators. However, when estimating the marginal variance, the BM estimator outperforms the others and the properties of the QML-*t* and BQML-*t* estimators are similar. Consequently, the BQML-*t* estimator seems to be better than QML-*t* one in estimating each of the parameters individually but not in estimating the unconditional variance.

Fig. 2 plots the Monte Carlo means of ξ_t. In this case, the three robust methods have smaller biases than ML. Furthermore, the mean errors corresponding to the periods of time around the outlier appearance are clearly smaller when using $r_c(x)$ defined in (6) instead of (5). Also notice that, although BQML-*t* is better than QML-*t* in estimating the parameters when there is a single outlier, the reverse is true in estimating the volatilities. One possible explanation is that the QML-*t* estimator seems to estimate better the marginal variance. We tend to think that good parameter estimates lead to good volatility estimates; see Charles and Darné (2006). However, this might not be true due to non-linearities. For example, consider a GARCH(1, 1) model with parameters $\theta = (\alpha_0, \alpha_1, \beta) = (0.1, 0.1, 0.8)$ and marginal variance 1. Suppose we have two parameter estimates: $\tilde{\theta}_1 = (0.02, 0.08, 0.9)$ and $\tilde{\theta}_2 = (0.11, 0.099, 0.85)$. Clearly, $\tilde{\theta}_2$ is closer to the true θ. However, estimates given by $\tilde{\theta}_1$ are closer to the true marginal variance. Therefore, it is expected that $\tilde{\theta}_1$ leads to better volatility estimates in spite of not being as close to θ as $\tilde{\theta}_2$.

Finally, we have repeated the analysis for Volatility Outliers finding similar results. They are available from the authors upon request.

4. Empirical application

Daily returns of the S&P 500 observed from January 2, 1987 to February 19, 2008 are analyzed to illustrate the differences in the alternative volatility estimates. Fig. 3 plots the returns in which volatility clustering and outliers seem to be present. Table 2, which reports ML estimates of the GARCH(1, 1) parameters, shows that the estimated marginal variance is $\widehat{\alpha}_0^{\text{ML}}/(1 - \widehat{\alpha}_1^{\text{ML}} - \widehat{\beta}^{\text{ML}}) = 1.45$.

QML-*t*, BQML-*t* and BM estimates are also reported in Table 2. As we can see, according to our simulation results, the estimates of α_0 and α_1 are larger while β is smaller when using ML compared to robust methods. For this particular series, BQML-*t* and QML-*t* give exactly the same values and therefore just one is considered.

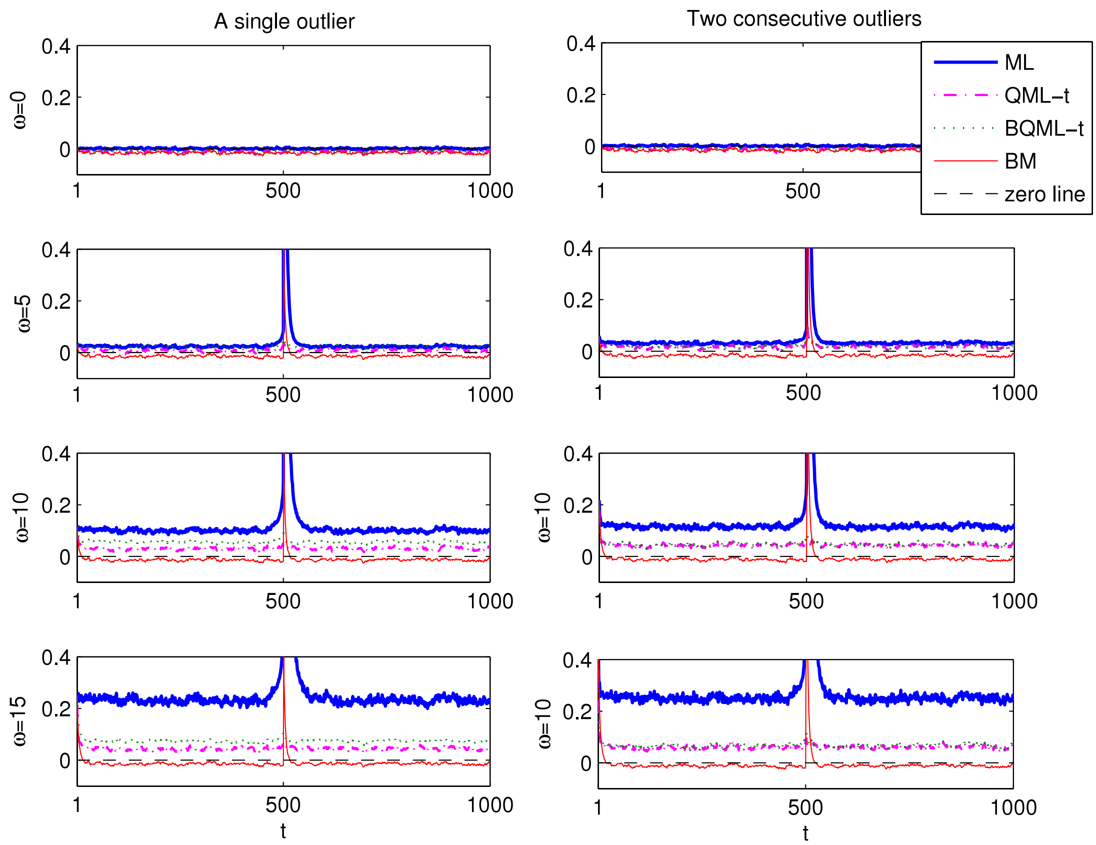

Fig. 2. Monte Carlo means of ξ_t in the presence of a single and two consecutive outliers in GARCH(1, 1) series of size $T = 1000$.

Table 2
Estimated parameters of the GARCH(1, 1) model fitted to S&P 500 daily returns together with their asymptotic standard deviations.

	α_0	α_1	β	η
ML	0.0141 (0.001)	0.0856 (0.002)	0.9047 (0.004)	–
QML-*t*	0.0058 (0.002)	0.0540 (0.008)	0.9416 (0.008)	0.1588 (0.003)
BQML-*t*	0.0058 (0.002)	0.0540 (0.008)	0.9416 (0.008)	0.1588 (0.003)
BM	0.0051	0.0559	0.9345	–

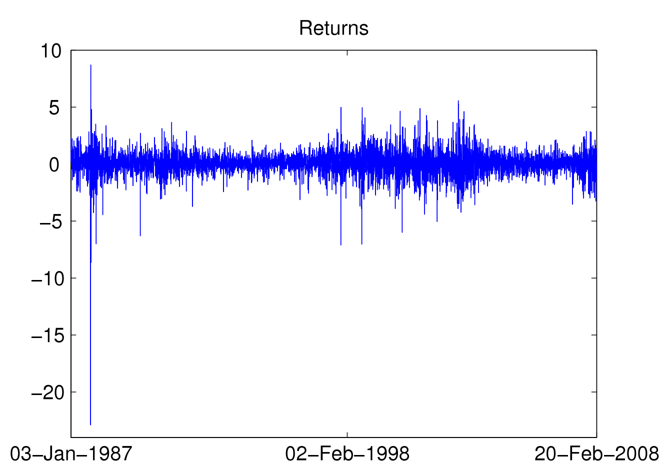

Fig. 3. Daily S&P 500 returns observed from January 2, 1987 to February 19, 2008.

On the other hand, the BM and BQML-*t* estimates of α_1 are similar, with the BM estimate slightly larger. The estimated persistence is close to one and similar for all estimators considered. Moreover, the BQML-*t* and BM estimates of the marginal variance are 1.32 and 0.53 respectively, smaller than the ML estimate. Finally, the BQML-*t* estimated degrees of freedom is 6.30, suggesting a heavy tailed distribution which can be attributed to the presence of outliers.

Estimated GARCH volatilities are plotted in Fig. 4. The main diagonal of the picture contains $\widehat{\sigma}_t^2$, computed using ML, BQML-*t* and BM estimators. The plots above the diagonal are scatter plots of the estimated volatilities. For example, the graph in the first row and third column is the scatter plot of $(\widehat{\sigma}_t^{\mathrm{ML}})^2$ against $(\widehat{\sigma}_t^{\mathrm{BM}})^2$. The plots below the diagonal contain the difference between estimated conditional variances. For example, the graph in the second row and first column plots $(\widehat{\sigma}_t^{\mathrm{BQML}\text{-}t})^2$-$(\widehat{\sigma}_t^{\mathrm{ML}})^2$. Clearly, ML tends to estimate larger volatilities compared to robust methods. By looking at the scatter plots in the first row, we can see that most of the points are above the 45° line, meaning that robust volatility estimates are smaller than $(\widehat{\sigma}_t^{\mathrm{ML}})^2$. The same conclusion can be obtained by looking at the plots in the first column. Most of the values are negative, meaning that $(\widehat{\sigma}_t^{\mathrm{ML}})^2$ is larger than any of the two robust estimates, which are both very similar.

The systematic differences observed when estimating the volatility by ML using expression (3) may have important implications on real financial applications. For example, when estimating the VaR, the uncertainty associated with returns will be larger than that obtained when estimating the volatility by

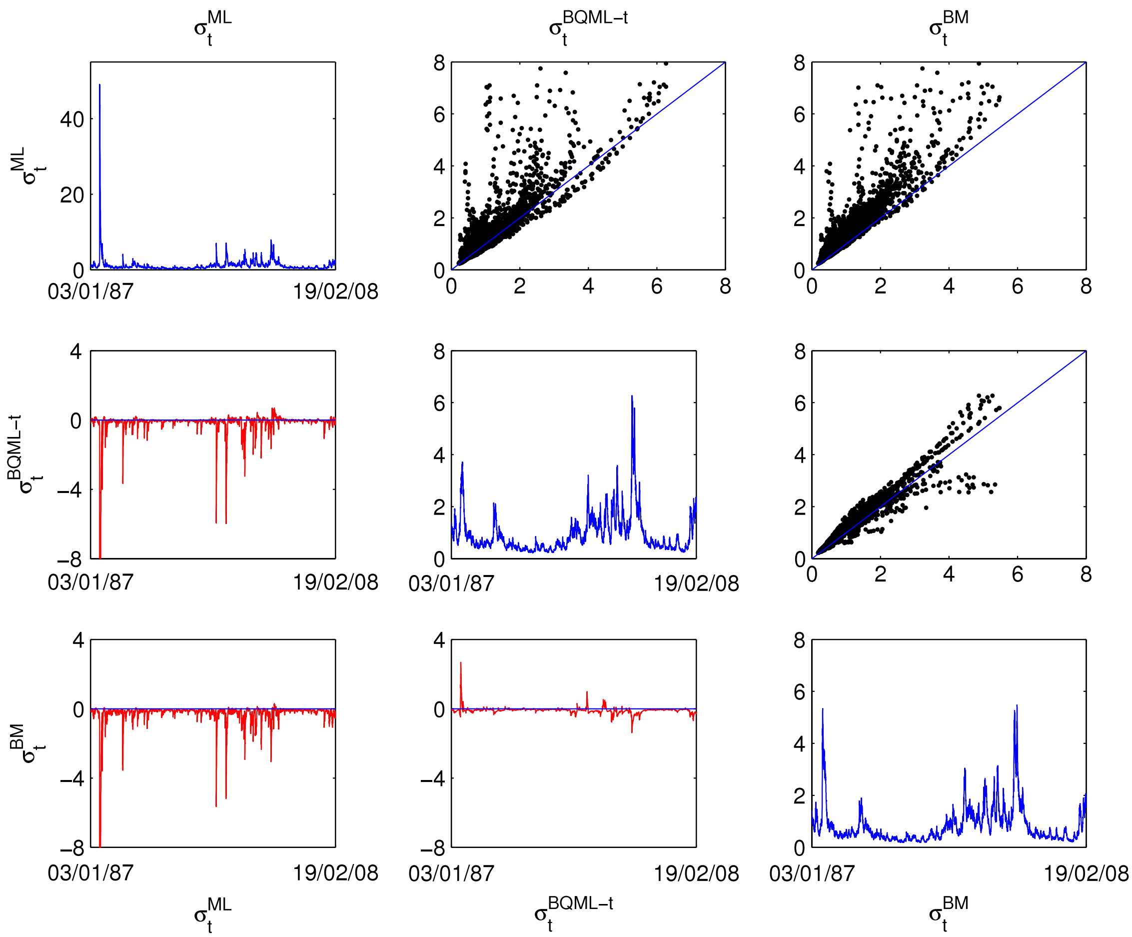

Fig. 4. Estimated volatilities $\hat{\sigma}_t^2$, scatter-plot of different estimated volatilities and difference between estimated volatilities for the S&P 500.

robust methods and, consequently, it will seem that the risk is larger. Larger risk is associated with larger capital requirements and, therefore, with a loss of inversion opportunities.

5. Conclusions

When outliers are present, biased estimators of the GARCH parameters lead to biases in the estimated volatilities in a non-linear way, so small biases in the estimated parameters do not guarantee small biases in estimated volatilities. Our results suggest that using robust procedures is a good strategy and the best performance is achieved when estimating the parameters by the BM estimator and filtering the volatilities by substituting large observations by their conditional standard deviations.

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