On cointegration for processes integrated at different frequencies (Periodic Polynomial Cointegration)

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Febrary 2020

Abstract

This paper explores the possibility of having cointegration relationships between processes integrated at different frequencies. We found that it is possible to establish this kind of cointegration and that the only possible cointegration relationships between processes integrated at different frequencies is full periodic polynomial cointegration. We explore the connection of this kind of cointegration with the demodulator operator and finally propose a simple way to test for the presence of cointegration between processes integrated at different frequencies based on the use of the demodulator operator.

Keywords: Periodic Cointegration, Polynomial cointegration, Demodulator Operator.

JEL codes: C32.

1 Introduction

To date, the vast literature on cointegration has focused primarily on the long-run characteristics of economic time series through the analysis of zero frequency unit roots. Nevertheless, economic and financial time series may exhibit unit roots at other frequencies; in particular, Engle, Granger, Hylleberg and Lee (1993), Johansen and Schaumburg (1999), Ahn and Reinsel (1994) and Bauer and Wagner (2012) analyze the seasonal case, while Bierens (2001) and Caporale, Cuñado and Gil-Alana (2013) consider unit roots associated with the business cycle. However, such analyses typically examine a specific frequency, without allowing the possibility that the responses of economic agents may vary over the seasonal or business cycle. In contrast, this paper studies long-run linkages between time series with unit roots at different frequencies and, specifically, the nature of any cointegration between a conventional zero frequency $I(1)$ series, denoted $I_0(1)$, and a series with a unit root at a business cycle or seasonal frequency. To our knowledge, no previous study has examined the nature of any such cointegration. Succinctly stating our main result, we show that cointegration can exist between time series that are integrated at different frequencies, with this being a specific type of time-varying polynomial cointegration. More specifically, the cointegrating relationship is dynamic with coefficients that exhibit cyclical variation, so that (for example) a long-run relationship can vary over the business cycle.

Polynomial cointegration is discussed in the literature in the contexts of (so-called) seasonal cointegration and multicointegration (see Hylleberg, Engle, Granger and Yoo, 1990, and Granger and Lee, 1989, respectively), while Gregoir (1999a, 1999b) undertakes a general analysis of these cases. Cubadda (2001) provides an alternative representation of the polynomial cointegration arising in the seasonal case in terms of complex-valued cointegration, which is developed further by Gregoir (2006, 2010). Although we take a similar approach to these latter authors, we relax the restrictions that cointegration applies only at a single frequency and that cointegrating vectors are time invariant.

As examined by Park and Hahn (1999) and Bierens and Martin (2010), time-varying cointegration allows the relevant coefficients to change smoothly over time in any direction. Such a general specification is, however, problematic in that it raises the question of what underlying mechanism drives these changes and hence it is not surprising that other authors place some economic structure on the nature of the temporal variation exhibited by the long-run relationship. For example, Hall, Psaradakis and Sola (1997) allow the cointegrating relationship to change with the economic environment through the use of a Markov-switching specification, while Birchenhall, Bladen-Hovell, Chui, Osborn and Smith (1989) apply periodic cointegration, in which the long-run coefficients vary with the time of the year.

The present paper generalizes periodic cointegration to show that temporal variation in the coefficients of a long-run relationship, with this variation being of a cyclic nature, can deliver cointegration between variables that are individually integrated at different frequencies. This approach encompasses not only variation associated with the seasons, but also over a cycle at a business cycle frequency, and the approach provides a specific form of regime-switching cointegration. Some of our results are implicit in analyses of periodic cointegration (see, in particular, Ghysels and Osborn, 2001, and Franses and Paap, 2004), but the cross-frequency cointegration implications have not previously been drawn out.

In our analysis, a central role is played by the complex demodulation operator, which transforms a real valued process integrated at a frequency different from zero to a complex valued process that is integrated at frequency zero. The idea of complex demodulation has long history in time series analysis (see, e.g., chapter 6 of Bloomfield (1976) and the references therein) but, to the best of our knowledge, it has not been previously used to investigate the presence of cointegration among series that are integrated at different frequencies.

This paper is organized as follows. Section 2 reviews the notions of integration at a given frequency. Section 3 presents our theoretical results. First, we show that two complex valued processes integrated at different frequencies can cointegrated and the connection of this fact with the demodulator operator. Second, we examine in details the various forms of cointegration that may exist between real valued time series integrated at different frequencies. Third, we tackle inferential issues. In Section 4, a Monte Carlo simulation exercise documents the small sample properties of the tests that we suggest. Section 5 presents an empirical application to illustrate concepts and methods. Finally, Section 6 concludes.

2 Integration at a frequency

For what follows, it will be useful to have a notation for the operator that removes a single unit root at a spectral frequency $\omega \in [0, \pi]$. To this end, and following Gregoir (1999a) and Cubadda (1999), we adopt the notation

$$\Delta_\omega = \begin{cases} 1 - e^{-i\omega}L, & \omega = 0,\ \pi \\ 1 - 2\cos\omega L + L^2 = (1 - e^{-i\omega}L)(1 - e^{i\omega}L), & \omega \in (0, \pi) \end{cases} \tag{1}$$

where L is the conventional lag operator. Special cases of this operator include:

$$\begin{aligned} \Delta_0 &= 1 - L \\ \Delta_\pi &= 1 + L \\ \Delta_{\pi/2} &= 1 + L^2 \\ \Delta_{\pi/3} &= 1 - L + L^2 \end{aligned}$$

so that Δ_0 is the conventional first difference operator, Δ_π and $\Delta_{\pi/2}$ are the operators that remove unit roots at the semi-annual and annual frequencies, respectively, for a seasonally integrated quarterly process (Hylleberg *et al.*, 1990), while $\Delta_{\pi/3}$ will remove a unit root corresponding to a cycle of six years duration in annual data.

To pin down the concept of integration at some frequency ω, we adopt the following definition, used by Gregoir (1999a):

Definition 1. *A purely nondeterministic real-valued random process x_t is integrated of order d, for non-negative integer d, at frequency $\omega \in [0, \pi]$ if $\Delta_\omega^d x_t$ is a covariance stationary process such that, for zero mean white noise ε_t, its Wold representation*

$$\Delta_\omega^d x_t = c(L)\varepsilon_t = \sum_{i=0}^{\infty} c_i \varepsilon_{t-i}$$

satisfies $\sum_{i=0}^{\infty} c_i^2 < \infty$ and $c(e^{i\omega}) \neq 0$.

Following Hylleberg *et al.* (1990) and Gregoir (1999a), a process x_t satisfying Definition 1 is denoted

$$x_t \sim I_\omega(d).$$

Although Gil-Alana (2001) and Gray, Zhang and Woodward (1989) allow fractional d, which is particularly relevant for financial time series, we are interested in cointegration for unit root economic time series which are typically $I_\omega(1)$ after taking account of deterministic effects. Obviously, $x_t \sim I_0(1)$ corresponds to a conventional (single) unit root process integrated at the zero frequency.

Although the differencing operator Δ_ω of (1) is defined for a real valued series, it is useful for the analysis that follows to consider complex valued processes. Specifically, when $x_t \sim I_\omega(1)$ we consider individually each of the two factors $\left(1 - e^{\pm i\omega} L\right)$ of Δ_ω for $\omega \in (0, \pi)$. Then, for

$$x_t = (2\cos\omega)\, x_{t-1} - x_{t-2} + \nu_t, \tag{2}$$

with $\nu_t \sim I_\omega(0)$ real valued, define the complex valued process

$$x_t^- = x_t - e^{i\omega} x_{t-1}. \tag{3}$$

It is then straightforward to see that

$$x_t^- = e^{-i\omega} x_{t-1}^- + \nu_t. \tag{4}$$

Successive substitution from (4) yields

$$\begin{aligned} x_t^- &= e^{-i\omega t} x_0^- + \sum_{j=0}^{t-1} e^{-i\omega j} v_{t-j} \\ &= e^{-i\omega t} x_0^- + \sum_{k=1}^{t} e^{-i\omega(t-k)} v_k \\ &= e^{-i\omega t} \left[x_0^- + \sum_{k=1}^{t} e^{i\omega k} v_k \right]. \end{aligned} \tag{5}$$

As noted by Gregoir (1999a, 2006) and del Barrio Castro, Rodrigues and Taylor (2017a,2017b), (5) is equivalent to:

$$x_t^- = e^{-i\omega t} \left[x_0^- + \sum_{k=1}^{t} e^{i\omega k} v_k \right] = e^{-i\omega t} x_t^{(0)-}. \tag{6}$$

the complex valued process $x_t^{(0)-} = x_0^- + \sum_{k=1}^{t} e^{i\omega k} \nu_k$ is integrated of order one at the zero frequency $(I_0(1))$[1], while $e^{-i\omega t}$ is the demodulation operator that shifts the zero frequency peak to frequency ω, leading to a complex valued $I_\omega(1)$ process. The demodulation operator provides the key to cointegration between processes integrated at different frequencies, examined in subsequent sections[2].

Using an analogous line of argument, it can be seen that this real valued process is also implied by the complex valued process

$$\begin{aligned} x_t^+ &= e^{i\omega} x_{t-1}^+ + v_t, \\ x_t^+ &= e^{i\omega t} x_t^{(0)+} \\ x_t^{(0)+} &= x_0^- + \sum_{k=1}^{t} e^{-i\omega k} \nu_k \end{aligned} \tag{7}$$

which forms a complex conjugate pair with x_t^-.

In the appendix it could be found Lemma 1 where the stochastic characteristics of 4) are summarized. To do that, we use the vector of seasons representation of (4) and also we use the double subscript notation (see the appendix for a detailed explanation of the use of double subscript notation $x_{n\tau}$) Also in Lemma 1 we present the results for $\omega_j = 2\pi j/N$ with $j = 1, \ldots, (N-1)/2$, that completes a full cycle every N/j periods.

Finally note that for (5) $x_t^- = e^{-i\omega_j t} x_t^{(0)-}$ and (7) $x_t^+ = e^{i\omega_j t} x_t^{(0)+}$ it is possible to write $x_t^{(0)-} = e^{i\omega_j t} x_t^-$ and $x_t^{(0)+} = e^{-i\omega_j t} x_t^+$ respectively.

[1] Note that as $\nu_t \sim I_\omega(0)$ acts as the innovation for the real valued $I_\omega(1)$ process (2), $e^{i\omega k} v_k$ is the complex valued innovation for $x_t^{(0)-}$.

[2] The demodulator operator it is also used in Theorem 4 of Johansen and Schaumburg (1999) in a multivariant setting.

3 Cointegration between processes integrated at different frequencies

In this section we initially focus on the long run relationships between complex valued process associated to different frequencies and show that it is possible to establish long-run relationships between complex valued integrated processes associated to different frequencies and that this cointegration is periodic. In the second subsection we show that between real valued processes the long run relationship between processes integrated at different frequencies should be periodic and polynomial. As we will work with two processes associated to different frequencies, hence we will define $\omega_j = 2\pi j/N$ and $\omega_k = 2\pi k/N$ with $j \neq k$.

3.1 Cointegration between complex $I_{\omega_j}(1)$ and $I_{\omega_k}(1)$ processes

Based on the results of the previous section, it is possible to define the following triangular system of a long run relationship (cointegration) between two complex valued process associated to the zero frequency:

$$\begin{aligned} y_t^{(0)-} &= \beta x_t^{(0)-} + u_t \\ x_t^{(0)-} &= x_{t-1}^{(0)-} + e^{t i\omega_j} \nu_{n\tau} \end{aligned} \tag{8}$$

where both the parameter of the long run relationship β and the $I(0)$ innovation u_t are complex valued. Note that if we multiply by $e^{-i\omega_j t}$ both sides $y_t^{(0)-} = \beta x_t^{(0)-} + u_t$ we obtain the triangular system of Gregoir (2010) (pp 1500):

$$\begin{aligned} e^{-i\omega_j t} y_t^{(0)-} &= \beta e^{-i\omega_j t} x_t^{(0)-} + e^{-i\omega_j t} u_t \\ e^{-i\omega_j t} x_t^{(0)-} &= x_t^{-} \\ x_t^{-} &= e^{-i\omega_j} x_{t-1}^{-} + \nu_t. \end{aligned} \tag{9}$$

Note also that it is possible to write $x_t^{(0)-} = e^{i\omega_j t} x_t^{-}$, hence if we replace it in (8) we get:

$$\begin{aligned} y_t^{(0)-} &= \beta e^{i\omega_j t} x_t^{-} + u_t \\ x_t^{-} &= e^{-i\omega_j} x_{t-1}^{-} + \nu_t \end{aligned} \tag{10}$$

Hence now we have a long-run relationship between $y_t^{(0)-}$ a complex valued integrated process at the zero frequency $I_0(1)$ and x_t^{-} a complex valued integrated process at the frequency ω_j, $I_{\omega_j}(0)$. It is also important to note that between $y_t^{(0)-}$ and x_t^{-} we have a periodic cointegration, as the cointegration vector varies with the seasons[3] $\beta e^{i\omega_j t} = \beta e^{i\omega_j (N(\tau-1)+n)} = \beta e^{i\omega_j n}$. Lemma 5 in the appendix summarize the stochastic behaviour. of the triangular system (10) using bi-variant vector of seasons for the triangular system defined in (10). It is clear from (34) that $y_t^{(0)-}$ and x_t^{-} share a common stochastic trend and hence that we have $N-1$ cointegration relationships between the seasons of the variables included in bi-variant vector of seasons.

Note also that if in (10) we premultiply by the demodulator operator $e^{-i\omega_k t}$ that will shift the complex valued zero frequency process $y_t^{(0)-}$ from the zero frequency to frequency $\omega_k = 2\pi k/N$ with $k = 0, 1, \ldots, \lfloor N/2 \rfloor$ and with $j \neq k$[4] we have:

$$e^{-i\omega_k t} y_t^{(0)-} = \beta e^{i[\omega_j - \omega_k]t} x_t^{-} + e^{-i\omega_k t} u_t.$$

Where we could define $y_t^{-} = e^{-i\omega_k t} y_t^{(0)-}$, clearly y_t^{-} is a complex valued process integrated at frequency ω_k sharing a common stochastic trend with x_t^{-}, that is a complex valued process integrated at frequency ω_j. Hence we have the following system with one $I_{\omega_j}(1)$ and one $I_{\omega_k}(1)$ with a periodic cointegration relationship $\left[1, -\beta e^{i[\omega_j - \omega_k]t}\right]'$, as we have different coefficient per each season:

$$\begin{aligned} y_t^{-} &= \beta e^{-i[\omega_k - \omega_j]t} x_t^{-} + e^{-i\omega_k t} u_t \\ x_t^{-} &= e^{-i\omega_j} x_{t-1}^{-} + \nu_t. \end{aligned} \tag{11}$$

In Lemma 6 in the appendix we summarize the stochastic behaviour. of the triangular system (11) using a bi-variant vector of seasons and double subscript notation.

[3] As $\omega_j = 2\pi j/N$ it is evident that $\omega_j\left(N(\tau-1)+n\right)$ is periodic and hene the identity $\beta e^{i\omega_j (N(\tau-1)+n)} = \beta e^{i\omega_j n}$ holds.
[4] If $j = k$ it is clear that we are in the case of (9).

Note also that if in (11) $e^{-i\omega_k t}$ is such that $\omega_{N/2} = \pi$ we finally have a long relationship between a process integrated at an harmonic frequency $\omega_j = 2\pi j/N$ and a process integrated at the Nyquist frequency (π).

To summarize with (10), (34), (11) and (43) we have shown that it is possible to define long-run relationship between complex valued integrated processes associated to different frequencies. In the next section we show that it is possible extend the situation to the case of real valued processes.

3.2 Cointegration between real valued $I_{\omega_j}(1)$ and $I_{\omega_k}(1)$ processes

In first place we are going to show that it is possible to establish a long-run relationship between a real valued process integrated at an harmonic frequency and a real valued process associated to the zero frequency. Based on the complex valued triangular representation (10) and in its complex conjugate system we have:

$$\begin{aligned} y_t^{(0)-} &= \beta e^{i\omega_j t} x_t^- + u_t \\ x_t^- &= e^{-i\omega_j} x_{t-1}^- + \nu_t \end{aligned} \tag{12}$$

$$\begin{aligned} y_t^{(0)+} &= \bar{\beta} e^{-i\omega_j t} x_t^+ + \bar{u}_t \\ x_t^+ &= e^{i\omega_j} x_{t-1}^+ + \nu_t \end{aligned} \tag{13}$$

where $\bar{u}_t$ and $\bar{\beta} = [\beta_R - i\beta_I]$ are the complex conjugates of u_t and $\beta = [\beta_R - i\beta_I]$ respectively. Note that x_t^- and x_t^+ are connected with a real valued process integrated at frequency ω_j say $\left(1 - 2\cos(\omega_j) L + L^2\right) x_t = \nu_t$ by $x_t^- = \left(1 - e^{i\omega_j} L\right) x_t$ and by $x_t^+ = \left(1 - e^{-i\omega_j} L\right) x_t$ respectively. Using this fact we get:

$$y_t^{(0)-} = [\beta_R + i\beta_I] e^{i\omega_j t} \left(1 - e^{i\omega_j} L\right) x_t + u_t \tag{14}$$

$$y_t^{(0)+} = [\beta_R - i\beta_I] e^{-i\omega_j t} \left(1 - e^{-i\omega_j} L\right) x_t + \bar{u}_t \tag{15}$$

$$\left(1 - 2\cos(\omega_j) L + L^2\right) x_t = \nu_t.$$

If we add both cointegration relationships (14) and (15) we obtain:

$$\begin{aligned} \left(y_t^{(0)-} + y_t^{(0)+}\right)/2 &= [\beta_R \cos(\omega_j t) - \beta_I \sin(\omega_j t)] x_t \\ &\quad - [\beta_R \cos(\omega_j [t+1]) - \beta_I \sin(\omega_j [t+1])] x_{t-1} \\ &\quad + (u_t + \bar{u}_t) \\ \left(1 - 2\cos(\omega_j) L + L^2\right) x_t &= \nu_t. \end{aligned} \tag{16}$$

Where $\left(y_t^{(0)-} + y_t^{(0)+}\right)/2$ is a real valued process associated to zero frequency $I_0(1)$, that shares the stochastic non-stationary behaviour of $\left(1 - 2\cos(\omega_j) L + L^2\right) x_t = \nu_t$ a real valued process $I_{\omega_j}(1)$ associated to frequency ω_j. Clearly in (16) we have periodic cointegration but also polynomial cointegration, note that as $t = N(\tau - 1) + n$ we could write:

$$\begin{aligned} [\beta_R \cos(\omega_j t) - \beta_I \sin(\omega_j t)] &= [\beta_R \cos(\omega_j n) - \beta_I \sin(\omega_j n)] = \beta_{1n} \\ [\beta_R \cos(\omega_j [t+1]) - \beta_I \sin(\omega_j [t+1])] &= [\beta_R \cos(\omega_j [n+1]) - \beta_I \sin(\omega_j [n+1])] = \beta_{2n}. \end{aligned} \tag{17}$$

In Lemma 8 in the appendix we summarize the stochastic behaviour of the triangular system (16).

In the case of long-run relationship between two real valued process integrated at different harmonic frequencies, based on the complex valued triangular representation (11) and its complex conjugate system[5]:

$$\begin{aligned} y_{n\tau}^- &= \beta e^{-i[\omega_k - \omega_j]n} x_{n\tau}^- + e^{-i\omega_k n} u_{n\tau} \\ x_{n\tau}^- &= e^{-i\omega_j} x_{n-1,\tau}^- + \nu_{n\tau} \end{aligned} \tag{18}$$

$$\begin{aligned} y_{n\tau}^+ &= \bar{\beta} e^{i[\omega_k - \omega_j]n} x_{n\tau}^+ + e^{i\omega_k n} \bar{u}_{n\tau} \\ x_{n\tau}^+ &= e^{i\omega_j} x_{n-1,\tau}^+ + \nu_{n\tau} \end{aligned} \tag{19}$$

[5] Using the double subscript notation described in the appendix , also note that as $t = N(\tau - 1) + n$, we have for example $e^{-i[\omega_k - \omega_j]t} = e^{-i[\omega_k - \omega_j](N(\tau-1)+n)} = e^{-i[\omega_k - \omega_j]n}$.

where as in the previous case β and $\bar{\beta}$ are complex conjugates and also $u_{n\tau}$ and $\bar{u}_{n\tau}$. Using the connection between the complex valued processes $y^-_{n\tau}$, $y^+_{n\tau}$, $x^-_{n\tau}$ and $x^+_{n\tau}$ and the real valued processes $y_{n\tau}$ and $x_{n\tau}$ it is possible to write:

$$\begin{aligned}
\left(1-e^{i\omega_k}L\right)y_{n\tau} &= \left[\beta_R+i\beta_I\right]e^{-i[\omega_k-\omega_j]n}\left(1-e^{i\omega_j}L\right)x_{n\tau}+e^{-i\omega_k n}u_{n\tau} &(20)\\
\left(1-e^{-i\omega_k}L\right)y_{n\tau} &= \left[\beta_R-i\beta_I\right]e^{i[\omega_k-\omega_j]n}\left(1-e^{-i\omega_j}L\right)x_{n\tau}+e^{i\omega_k n}\bar{u}_{n\tau} &(21)\\
\left(1-2\cos\left(\omega_j\right)L+L^2\right)x_{n\tau} &= \nu_{n\tau}.
\end{aligned}$$

by adding and subtracting both cointegration relationships we obtain:

$$\begin{aligned}
y_{n\tau} &= \cos\left(\omega_k\right)y_{n-1,\tau}+\left[\beta_R\cos\left(\left[\omega_k-\omega_j\right]n\right)+\beta_I\sin\left[\omega_k-\omega_j\right]n\right]x_{n\tau}\\
&\quad-\left[\beta_R\cos\left(\omega_k n-\omega_j\left[n+1\right]\right)+\beta_I\sin\left(\omega_k n-\omega_j\left[n+1\right]\right)\right]x_{n-1.\tau} \qquad (22)\\
&\quad+\left(e^{-i\omega_k n}u_{n\tau}+e^{i\omega_k n}\bar{u}_{n\tau}\right)/2
\end{aligned}$$

$$\begin{aligned}
\sin\left(\omega_k\right)y_{n-1,\tau} &= \left[\beta_R\sin\left(\left[\omega_k-\omega_j\right]n\right)-\beta_I\cos\left(\left[\omega_k-\omega_j\right]n\right)\right]x_{n\tau}\\
&\quad-\left[\beta_R\sin\left(\omega_k n-\omega_j\left[n+1\right]\right)-\right. \qquad (23)\\
&\quad\left.\beta_I\cos\left(\omega_k n-\omega_j\left[n+1\right]\right)\right]x_{n-1,\tau}\\
&\quad+\left(e^{-i\omega_k n}u_{n\tau}-e^{i\omega_k n}\bar{u}_{n\tau}\right)/\left(-2i\right).
\end{aligned}$$

Hence we have two cointegration relationships as in Gregoir (2010). In the case of $\omega_k=\omega_j$ (22)-(23) reduce to the result reported in Gregoir (2010). In our case we also have polynomial cointegration but also periodic cointegration.

For our convenience we summarize (22)-(23), in one expression using (22)+$\cos(\omega_k)/\sin(\omega_k)$(23), and obtain the following triangular system:

$$\begin{aligned}
y_{n\tau} &= \left[\beta_R\left(\cos\left(\left[\omega_k-\omega_j\right]n\right)+\frac{\cos\left(\omega_k\right)}{\sin\left(\omega_k\right)}\sin\left(\left[\omega_k-\omega_j\right]n\right)\right)-\beta_I\left(\sin\left(\left[\omega_k-\omega_j\right]n\right)-\frac{\cos\left(\omega_k\right)}{\sin\left(\omega_k\right)}\cos\left(\left[\omega_k-\omega_j\right]n\right)\right)\right]x_{n\tau}\\
&\quad-\left[\beta_R\left(\cos\left(\omega_k n-\omega_j\left[n+1\right]\right)+\frac{\cos\left(\omega_k\right)}{\sin\left(\omega_k\right)}\sin\left(\omega_k n-\omega_j\left[n+1\right]\right)\right)\right. \qquad (24)\\
&\quad\left.-\beta_I\left(\sin\left(\omega_k n-\omega_j\left[n+1\right]\right)-\frac{\cos\left(\omega_k\right)}{\sin\left(\omega_k\right)}\cos\left(\omega_k n-\omega_j\left[n+1\right]\right)\right)\right]x_{n-1,\tau}+\varpi_{n\tau}
\end{aligned}$$

$$\left(1-2\cos\left(\omega_j\right)L+L^2\right)x_t=\nu_t.$$

In Lemma 9 of the appendix we summarized the stochastic behaviour of the triangular system (24).

3.3 Econometric strategy

Once we have shown that it is possible to find long-run relationship between processes integrated at different frequencies. We need an Econometric strategy to detect this situation. To do that, we propose two different ways based on the Johansen (1996) procedure to determine the cointegration rank in a VAR model. Our first and simpler approach consist in testing the cointegration rank in a VAR model applied to a system of $2N$ variables given by vector of seasons representation of each of the two individual time series[6], which treats the intra-cycle $n=1,...,N$ as distinct time series. This approach is discussed by, for example, Ghysels and Osborn (2001, Chapter 6) in the context of contemporaneous periodic cointegration. However, it has the obvious disadvantage of typically requiring the use of high dimensional systems. For example if we have a two integrated time series one associated to the zero frequency and another associated to an harmonic frequency, if we do not have cointegration we should find 3 common trends or $2N-3$ cointegration relationships between the elements of the $2N\times 1$ vector formed with the vector of seasons of the two time series. But if we have cointegration, that is, Periodic Polynomial Cointegration, we should find evidence in favour of the existence of 2 common trends or $2N-2$ cointegration relationships between the elements of the $2N\times 1$ system. The basic problem of this first approach is that is going to be quite overparametrizised and it is also well known that the Johansen procedure do not works well when we have VAR models with a high dimension.

[6] In the case of k variables the system will be of kN where each of the time series is treated as $N\times 1$ vector of seasons.

The second approach to test for the presence of cyclic cointegration between $y_t \sim I_0(1)$ and $x_t \sim I_\omega(1)$ is to apply the transformation $e^{-i\omega t}(1-e^{-i\omega}L)x_t$ and $e^{i\omega t}(1-e^{i\omega}L)x_t$to the latter. Then using software that allows computations with complex variables, a test for cointegration is conducted in the bivariate system consisting of y_t and either $e^{-i\omega t}(1-e^{-i\omega}L)x_t$ or $e^{i\omega t}(1-e^{i\omega}L)x_t$. In the case of two real value processes associated to harmonic frequencies $y_t \sim I_{\omega_j}(1)$ and $x_t \sim I_{\omega_k}(1)$ is to apply the transformations $e^{-i\omega_j t}(1-e^{-i\omega_j}L)y_t$ and $e^{-i\omega_k t}(1-e^{-i\omega_k}L)x_t$ and test for cointegration between then and proceed in a equivalent way with $e^{i\omega_j t}(1-e^{i\omega_j}L)y_t$ and $e^{i\omega_k t}(1-e^{i\omega_k}L)x_t$. We apply the method proposed by Cubadda (2001) to test for cointegration in harmonic frequencies. Hence in this case we use the critical values collected in the paper by Cubadda (2001) as in our case the distribution of the test should be the same as the one reported by Cubadda (2001) in theorem 2.1. Note that in the Lemmas in the appendix we always have complex Brownian Motions as in theorem 2.1.

Finally the effect of the demodulator operator $e^{-i\omega_j t}$ on the deterministic part could be easily understand following the lines of chapter 7 in Bloomfield (2000). Note that for example seasonal dummies have a one to one correspondence with their trigonometric representation in terms of $\cos(\omega_j t)$ and $\sin(\omega_j t)$ note that $\cos(\omega_j t) = e^{-i\omega_j t} + e^{i\omega_j t}$, and hence $(1-e^{-i\omega_k}L)\cos(\omega_j t) = e^{i\omega_j t} - e^{i\omega_j (t-2)}$ and finally $e^{-i\omega_k t}(1-e^{-i\omega_k}L)\cos(\omega_j t) = 1 - e^{-i\omega_j 2}$ hence becomes a constant once demodulated.

4 Monte Carlo

In this section we undertake a Monte Carlo experiment to illustrate the nature of long-run relationships between processes that are nonstationary at different frequencies and also show how to test for cointegration in such situations. In particular consider examples with $N = 6$, and for two sample size of 100 and 200 complete cycles, and hence the total number of observations is $6T = 600$ and $6T = 1200$, respectively, for the two cases. The Monte Carlo results are based on 5,000 replications.

For the cointegration situation we work with two cases

Case I: Where we reproduce situation collected in (16) with $\omega_j = 2\pi/3$ and $\pi/3$. Hence we are in a situation of cointegration between an $I_0(1)$ process and an $I_{\omega_j}(1)$ process. Table 1 collects the results (16) with $\omega_j = \pi/3$, $\beta_R = \cos(-\pi/3)$, $\beta_I = \cos(-\pi/3)$ and $T = 200$. Clearly the results of table 1 show that it is possible to establish a long-run relationship between a process integrated in the frequency $\pi/3$ and a process integrated in the zero. For the sample size of $T = 200$ both method proposed in the previous section perform well. In table 2 we also collects the results from (16) but in this case for $\omega_j = 2\pi/3$, $\beta_R = \cos(-2\pi/3)$, $\beta_I = \cos(-2\pi/3)$ and $T = 200$ and 100. The results of table 2, also show that it is possible to establish cointegration between processes integrated at the zero and $2\pi/3$, but it also clearly show a poor performance of the vector of seasons approach with the sample size of $T = 100$ but the one based on the demodulator operator performs equally well for both sample sizes.

Case II: Where we reproduce situation collected in (24) with $\omega_k = 2\pi/3$ and $\omega_j = \pi/3$, $\beta_R = \cos(-\pi/3)$, $\beta_I = \cos(-\pi/3)$, $T = 200$ and 100 As in case I we could appreciate that with $T = 200$ both method perform well but with $T = 100$ only the demodulator approach perform properly.

Finally, in order to complete the picture, we also show the performance of the previous tests in situations when we do not have cointegration between the processes integrated at different frequencies. In particular we pay attention to the following cases:

$$\begin{aligned} x_{n\tau} &= 2\cos\left(2\frac{\pi}{3}\right)x_{n-1,\tau} - x_{n-2,\tau} + \varepsilon^x_{n\tau} \qquad \varepsilon^x_{n\tau}\,\tilde{}\,Niid(0,1) \\ y_{n\tau} &= y_{n-1,\tau} + \varepsilon^y_{n\tau} \qquad \varepsilon^y_{n\tau}\,\tilde{}\,Niid(0,1) \end{aligned} \tag{25}$$

$$\begin{aligned} x_{n\tau} &= 2\cos\left(2\frac{\pi}{3}\right)x_{n-1,\tau} - x_{n-2,\tau} + \varepsilon^x_{n\tau} \qquad \varepsilon^x_{n\tau}\,\tilde{}\,Niid(0,1) \\ y_{n\tau} &= 2\cos\left(\frac{\pi}{3}\right)y_{n-1,\tau} - y_{n-2,\tau} + \varepsilon^y_{n\tau} \qquad \varepsilon^y_{n\tau}\,\tilde{}\,Niid(0,1) \end{aligned} \tag{26}$$

With $T = 200$, the results are collected in table 4, and show that the two methods proposed work properly for a sample size of $T = 200$.

Despite that both methods proposed in the previous section to detect the presence of cointegration between processes integrated at different frequencies when the sample size is big, but the vector of seasons approach do not work so well as the sample size decreases, hence we advice to use the approach based on the demodulator operator.

5 Empirical application

In this section we explore the presence of cointegration relationship between processes integrated at different frequencies using data quarterly data from the Balearic Islands, in particular we analyze the connection between total employment $(emp_{n\tau})$ and tourist arrivals $(arr_{n\tau})$form the first quarter of 1979 to the fourth quarter of 2015. Picture 1 and 2 show the time series in logs. Using the HEGY test we find clear evidence of the presence of a unit root at the zero frequency for $\ln(emp_{n\tau})$. In the case of $\ln(arr_{n\tau})$ we find clear evidence of a unit root at the zero frequency and a pair of complex unit roots at frequency $\pi/2$ and mild evidence of the presence of a unit root at the Nyquist frequency. We have explore the possibility of a long run relationship between $\ln(emp_{n\tau})$ and $(1-L)\ln(arr_{n\tau})$ and also $\ln(emp_{n\tau})$ and $(1-L)(1+L)\ln(arr_{n\tau})$. That is, we are going to check if the non-stationary behaviour associated to frequency $\pi/2$ present in the tourism arrivals could affect the non-stationary behaviour observed in the employment. The following table collect the results obtained using the demodulator approach, the results consider the inclusion of quarterly seasonal dummies:

	$\ln(emp_{n\tau})$ $(1-L)\ln(arr_{n\tau})$					$\ln(emp_{n\tau})$ $(1-L)(1+L)\ln(arr_{n\tau})$			
VAR order	$e^{i\frac{\pi}{2}t}$		$e^{-i\frac{\pi}{2}t}$		VAR order	$e^{i\frac{\pi}{2}t}$		$e^{-i\frac{\pi}{2}t}$	
	$q_0=2$	$q_0=1$	$q_0=2$	$q_0=1$		$q_0=2$	$q_0=1$	$q_0=2$	$q_0=1$
2	4,8658	123,9097***	4,8658	123,9097***	2	4.7300	130.1296***	4.7300	130.1296***
3	4,4071	95,9816***	4,4071	95,9816***	3	4.3086	101.9734***	4.3086	101.9734***
4	0,2953	104,1451***	0,2953	104,1451***	4	0.2436	111.3259***	0.2436	111.3259***
5	0,1190	107,1158***	0,1190	107,1158***	5	0.0447	118.5896***	0.0447	118.5896***
6	0,1641	113,9965***	0,1641	113,9965***	6	0.1865	124.5702***	0.1865	124.5702***
7	0,4588	117,1246***	0,4588	117,1246***	7	0.4064	125.7330***	0.4064	125.7330***
8	0,0590	127,2409***	0,0590	127,2409***	8	0.0253	127.1980***	0.0253	127.1980***

6 References

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7 Appendix

It will be useful to introduce a double subscript notation $x_{n\tau}$ where the subscripts $n\tau$ indicate the n^{th} observation within the τ^{th} cycle and the total number of observations per cycle N. As in the later sections we are going to work with processes integrated at different spectral frequencies it will we convenient to denote the frequency associated to process $x_{n\tau}$as $\omega_j = 2\pi j/N$ with $j = 0, 1, \ldots, \lfloor N/2 \rfloor$ and $\lfloor . \rfloor$ denotes de integer part. Hence for example $\omega_1 = 2\pi/N$ completes a full cycle every N observations. A double subscript notation when considering an $I_{\omega_j}(1)$, with (2) then written as

$$x_{n\tau} = (2\cos\omega_j)\, x_{n-1,\tau} - x_{n-2,\tau} + v_{n\tau}, \quad n = 1,\, 2, \cdots, N \tag{27}$$

where the subscripts $n\tau$ indicate the n^{th} observation within the τ^{th} cycle at frequency ω_j. When using the double subscript notation, it is understood that $y_{n-k,\tau} = y_{N-n+k,\tau-1}$ for $n-k \le 0$. Adopting the convention that $t = 1$ corresponds to $n = \tau = 1$, then $t = N(\tau - 1) + n$ provides the one-to-one mapping between the two notations.

Define the $N \times 1$ vector of seasons $X_\tau^- = \left[x_{1\tau}^-, x_{2\tau}^-, x_{3\tau}^-, \ldots, x_{N\tau}^-\right]'$ to (4) and present the following lemma where we summarize the stochastic characteristics of (4):

Lemma 1 *For $X_\tau^- = \left[x_{1\tau}^-, x_{2\tau}^-, x_{3\tau}^-, \ldots, x_{N\tau}^-\right]'$ with $x_{n\tau}^-$ $n = 1, 2, \cdots, N$ defined in (4) and with $v_{n\tau} \sim iid\left(0, \sigma^2\right)$ it is possible to write:*

$$\begin{aligned} \frac{1}{\sqrt{T}} X_{\lfloor Tr \rfloor}^- &\Rightarrow \sigma \mathbf{C}_j^- W(r) = \sigma \mathbf{v}_j^- \mathbf{v}_j^{+'} W^v(r) \\ &= \sigma (N/2)^{1/2} \mathbf{v}_j^- \left(w_R(r) + i w_I(r)\right) \end{aligned} \tag{28}$$

where $\mathbf{C}_j^-$ is circulant matrix of rank one $\mathbf{C}_j^- = \mathrm{Ci}\,rc\left[1, e^{-i\omega_j}, e^{-i2\omega_j}, \cdots, e^{-i(N-1)\omega_j}\right]$, and the vectors $\mathbf{v}_j^-$ and $\mathbf{v}_j^+$ are defined as $\mathbf{v}_j^- = \left[\begin{array}{ccccc} e^{-i\omega_j} & e^{-i2\omega_j} & e^{-i3\omega_j} & \cdots & e^{-iN\omega_j}\end{array}\right]'$ and as $\mathbf{v}_j^+ = \left[\begin{array}{ccccc} e^{i\omega_j} & e^{i2\omega_j} & e^{i3\omega_j} & \cdots & e^{iN\omega_j}\end{array}\right]$. $W^v(r) = \left[\begin{array}{cccc} W_1^v(r) & W_2^v(r) & \cdots & W_N^v(r)\end{array}\right]'$ is a $N \times 1$ vector Brownian motion and $w_R^v(r)$ and $w_I^v(r)$ are two scalar Brownian motions defined as $w_R^v(r) = (S/2)^{-1/2} \sum_{k=1}^N \cos(k\omega_j) W_k^v(r)$ and $w_I^v(r) = (S/2)^{-1/2} \sum_{k=1}^N \sin(k\omega_j) W_k^v(r)$ respectively.

Remark 2 *Note that the result in Lemma 1 above also applies to x_t^+ (7) and it is straightforward to see that for $X_\tau^+ = \left[x_{1\tau}^+, x_{2\tau}^+, x_{3\tau}^+, \ldots, x_{N\tau}^+\right]'$ it is possible to write $T^{1/2} X_{\lfloor Tr \rfloor}^+ \Rightarrow \sigma \mathbf{C}_j^+ W^v(r) = \sigma \mathbf{v}_j^+ \mathbf{v}_j^{-'} W^v(r) = \sigma (N/2)^{1/2} \mathbf{v}_j^- \left(w_R^v(r) - i w_I^v(r)\right)$ with $\mathbf{C}_j^+ = \mathrm{Ci}\,rc\left[1, e^{i\omega_j}, e^{i2\omega_j}, \cdots, e^{i(N-1)\omega_j}\right]$. Hence we have a pair of complex conjugate complex valued scalar Brownian motions $w_R^v(r) \pm i w_I^v(r)$ as in Gregoir (2010) (see pp.1494). Note also that in del Barrio Castro, Rodrigues and Taylor (2017) it is shown a similar result but considering complex valued Near-Integrated processes and also allowing for serial correlation in the innovations (see pp 9 (3.12) and (3.13)).*

Remark 3 *From (28) and (5) it is clear that $\left(w_R^v(r) + i w_I^v(r)\right)$ collects the behaviour. of the stochastic trend $\left[x_0^- + \sum_{k=1}^t e^{i\omega_j k} v_k\right]$ and the vector $\mathbf{v}_j^- = \left[\begin{array}{ccccc} e^{-i\omega_j} & e^{-i2\omega_j} & e^{-i3\omega_j} & \cdots & e^{-iN\omega_j}\end{array}\right]'$ the effect of the demodulator operator $e^{-it\omega_j}$. Another interesting point of (28) is that it clearly show that the seasons associated to X_τ^- share a common stochastic trend, or equivalently that between the seasons of X_τ^- we have $N-1$ cointegration relationships.*

Remark 4 *In the case of process $x_{n\tau}$ (27) as shown in Smith, Taylor and del Barrio Castro (2009) Lemma 1 and remark pp 540 we have the circulant matrix of rank 2 $\mathbf{C}_j = \mathrm{Ci}\,rc\left[\frac{\sin(\omega_j)}{\sin(\omega_j)}, \frac{\sin(N\omega_j)}{\sin(\omega_j)}, \frac{\sin([N-1]\omega_j)}{\sin(\omega_j)}, \ldots, \frac{\sin(2\omega_j)}{\sin(\omega_j)}\right]$ and note that we have the following relationship between $\mathbf{C}_j$, $\mathbf{C}_j^-$ and $\mathbf{C}_j^+$ $\mathbf{C}_j = \frac{e^{-i\omega_j}}{e^{-i\omega_j} - e^{i\omega_j}} \mathbf{C}_j^- + \frac{e^{i\omega_j}}{e^{i\omega_j} - e^{-i\omega_j}} \mathbf{C}_j^+$.*

Proof of Lemma 1: First note that the process (4) admits the following vector of seasons representation $X_\tau^- = \left[x_{1\tau}^-, x_{2\tau}^-, ..., x_{N\tau}^-\right]'$, then

$$\Phi_0^- X_\tau^- = \Phi_1^- X_{\tau-1}^- + V_\tau \tag{29}$$

where, $V_\tau = \left[v_{1\tau}^-, v_{2\tau}^-, ..., v_{N\tau}^-\right]'$ and Φ_0^- and $\Phi_1^-$are $N \times N$ matrices with the generic element of

$$\begin{aligned} \phi_{0(h,j)}^- &= \begin{cases} 1 & h = j,\ j = 1, ..., N \\ -e^{-i\omega} & h = j+1,\ j = 1, ..., N-1 \\ 0 & \text{otherwise} \end{cases} \\ \phi_{1(h,j)}^- &= \begin{cases} e^{-i\omega} & h = 1,\ j = N \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

in which the subscript (h,j) indicates the $(h,j)^{th}$ element of the respective matrix. As in the quarterly $PI(1)$ model studied by Paap and Franses (1999), successively substituting in (29) yields

$$\begin{aligned} X_\tau^- &= [(\Phi_0^-)^{-1}\Phi_1^-]^\tau X_0^- + (\Phi_0^-)^{-1}V_\tau + \sum_{j=1}^{\tau-1}[(\Phi_0^-)^{-1}\Phi_1^-]^j(\Phi_0^-)^{-1}V_{\tau-j} \\ &= [(\Phi_0^-)^{-1}\Phi_1^-]^\tau X_0^- + (\Phi_0^-)^{-1}V_\tau + [(\Phi_0^-)^{-1}\Phi_1^-](\Phi_0^-)^{-1}\sum_{j=1}^{\tau-1}V_{\tau-j}. \end{aligned} \tag{30}$$

Note that this result follows because $(\Phi_0^-)^{-1}\Phi_1^-$ is idempotent, which can be seen by generalizing the form of $(\Phi_0^-)^{-1}\Phi_1^-$ presented by Paap and Franses (1999) and noting that $e^{-i\omega N} = 1$ for $x_{n\tau}^- \sim I_\omega(1)$, and hence[7] $[(\Phi_0^-)^{-1}\Phi_1^-]^j = [(\Phi_0^-)^{-1}\Phi_1^-]$ for $j = 2, 3, ...$Clearly, (30) provides an alternative representation of (5), but now expressed in terms of the vector of (complex-valued) observations over an entire cycle, where $[(\Phi_0^-)^{-1}\Phi_1^-](\Phi_0^-)^{-1}$ gives the effect of the accumulated vector of shocks $\sum_{j=1}^{\tau-1}V_{\tau-j}$ (see for example Boswijk and Franses (1996), Paap and Franses (1999), del Barrio Castro and Osborn (2008), and for complex-valued processes in the context of seasonally integrated processes by del Barrio Castro, Rodrigues and Taylor (2017)). The matrix $[(\Phi_0^-)^{-1}\Phi_1^-](\Phi_0^-)^{-1}$ has rank one and hence we can write it as

$$\mathbf{C}_j^- = [(\Phi_0^-)^{-1}\Phi_1^-](\Phi_0^-)^{-1} = (\mathbf{a}_j^-)(\mathbf{b}_j^-)' \tag{31}$$

where, for (31),

$$\begin{aligned} \mathbf{a}_j^- &= \begin{bmatrix} 1 & e^{-i\omega_j} & e^{-i2\omega_j} & \cdots & e^{-i(N-1)\omega_j} \end{bmatrix}' \\ \mathbf{b}_j^- &= \begin{bmatrix} 1 & e^{-i(N-1)\omega_j} & e^{-i(N-2)\omega_j} & \cdots & e^{-i\omega_j} \end{bmatrix}'. \end{aligned} \tag{32}$$

Therefore, in (30), the scalar partial sum $(\mathbf{b}_j^-)'\sum_{j=1}^{\tau-1}V_{\tau-j}$ is integrated at the zero frequency, while $\mathbf{a}_j^-$ allocates this across the N observations of the cycle at frequency ω_j, yielding an $I_{\omega_j}(1)$ process. This is, of course, the same as in (5)-(6), but the cyclicality of the resulting process is now made clear by representing the demodulation operator through the vector $\mathbf{a}_j^-$. Note also that the $N \times N$ matrix $\mathbf{C}_j^-$ is a Circulant matrix $\mathbf{C}_j^- = \mathrm{Ci}\,rc\left[1, e^{-i\omega_j}, e^{-i2\omega_j}, \cdots, e^{-i(N-1)\omega_j}\right]$. Note also that we have $\frac{1}{\sqrt{T}}\sum_{j=1}^{\lfloor Tr\rfloor}V_j \Rightarrow \sigma W(r)$ where $W(r)$ is a $N \times 1$ vector Brownian motion. Also for the circulant matrix $\mathbf{C}_j^-$ it is possible to write:

$$\begin{aligned} \mathbf{C}_j^- &= (\mathbf{a}_j^-)(\mathbf{b}_j^-)' = (\mathbf{a}_j^-)e^{-i\omega_j}e^{i\omega_j}(\mathbf{b}_j^-)' = \mathbf{v}_j^-\mathbf{v}_j^{+\prime} \\ \mathbf{v}_j^- &= \begin{bmatrix} e^{-i\omega_j} & e^{-i2\omega_j} & e^{-i3\omega_j} & \cdots & e^{-iN\omega_j} \end{bmatrix}' \\ \mathbf{v}_j^+ &= \begin{bmatrix} e^{i\omega_j} & e^{i2\omega_j} & e^{i3\omega_j} & \cdots & e^{iN\omega_j} \end{bmatrix}'. \end{aligned} \tag{33}$$

Finally note that $\mathbf{v}_j^{+\prime}W(r) = \sum_{k=1}^N e^{ik\omega_j}W_k(r) = \sum_{k=1}^N \cos(k\omega_j)W_k(r) + i\sum_{k=1}^N \sin(k\omega_j)W_k(r)$, and that $w_R(r) = (S/2)^{-1/2}\sum_{k=1}^N \cos(k\omega_j)W_k(r)$ and $w_I(r) = (S/2)^{-1/2}\sum_{k=1}^N \sin(k\omega_j)W_k(r)$.■

Lemma 5 *For* $Z_\tau^{(0,\omega_j)-} = \left[y_{1\tau}^{(0)-}, y_{2\tau}^{(0)-}, \ldots, y_{N\tau}^{(0)-}, x_{1\tau}^-, x_{2\tau}^-, \ldots, x_{N\tau}^-\right]'$ *with* $x_{n\tau}^-$ *and* $y_{n\tau}^{(0)-}$ $n = 1, 2, \cdots, N$ *defined in (10) and with* $v_{n\tau} \sim iid\left(0, \sigma^2\right)$ *and* $u_{n\tau} \sim iid\left(0, \sigma_u^2\right)$ *it is possible to write:*

$$\frac{1}{\sqrt{T}}Z_{\lfloor Tr\rfloor}^{(0,\omega_j)-} \Rightarrow \sigma(N/2)^{1/2}\begin{bmatrix} \beta\mathbf{1} \\ \mathbf{v}_j^- \end{bmatrix}\left(w_R^v(r) + iw_I^v(r)\right) \tag{34}$$

with $\left(w_R^v(r) + iw_I^v(r)\right)$ *and* $\mathbf{v}_j^-$ *as in Lemma 1 and finally* $\mathbf{1}$ *is a* $N \times 1$ *vector of ones.*

Proof of Lemma 5: In the case of triangular system (10) we then define the vectors $Z_\tau^{(0,\omega_j)-} = \left[y_{1\tau}^{(0)-}, y_{2\tau}^{(0)-}, \cdots, y_{N\tau}^{(0)-}, x_{1\tau}^-, x_{2\tau}^-, \cdots, x_{N\tau}^-\right]'$ and $V_\tau^Z = [u_{1\tau}, u_{2\tau}, \cdots, u_{N\tau}, v_{1\tau}, v_{2\tau}, \cdots, v_{N\tau}]' = [V_\tau^{u\prime}V_\tau']'$ with

[7] Note that matrix Φ_0^- (see chapter 2 pp 45-48 of Pollock (1999))is a $N \times N$ lower-triangular Toeplitz matrix associated to the polynomial $\left(1 - e^{-i\omega_j}L\right)$. Hence in matrix $\left(\Phi_0^-\right)^{-1}$ it will be collected the coefficientes on the expansion of the inverse polynomial associated to $\left(1 - e^{-i\omega_j}L\right)$. Based on the form of matrices $(\Phi_0^-)^{-1}$ and Φ_1^-it is clear that he resulting matrix $(\Phi_0^-)^{-1}\Phi_1^-$ will be a $N \times N$ matrix with the firsts $N-1$ columns with elements equal to zero and the last column equal to the column vector $\mathbf{v}_j^- = \begin{bmatrix} e^{-i\omega_j} & e^{-i2\omega_j} & e^{-i3\omega_j} & \cdots & e^{-iN\omega_j} \end{bmatrix}'$. finally note that the last element of $\mathbf{v}_j^-$, that is, $e^{-iN\omega_j}$ is equal to 1 as $\omega_j = 2\pi j/N$ and hence the idempotency of $(\Phi_0^-)^{-1}\Phi_1^-$ could be deduced easily.

$V_\tau = [v_{1\tau}, v_{2\tau}, \cdots, v_{N\tau}]'$ as in Lemma 1 and $V_\tau^u = [u_{1\tau}, u_{2\tau}, \cdots, u_{N\tau}]'$, using the same line of argument as in the proof of Lemma 1 we can write

$$\Phi_0^- Z_\tau^{(0,\omega_j)-} = \Phi_1^- Z_{\tau-1}^{(0,\omega_j)-} + V_\tau^Z \tag{35}$$

where

$$\Phi_0^- = \begin{bmatrix} I_{N\times N} & \Phi_0^{-12} \\ 0_{N\times N} & \Phi_0^{-22} \end{bmatrix}, \quad \Phi_1^- = \begin{bmatrix} 0_{N\times N} & 0_{N\times N} \\ 0_{N\times N} & \Phi_1^{-22} \end{bmatrix}$$

in which all sub-matrix are $N \times N$, with

$$\begin{aligned} \Phi_0^{-12} &= Diag\left[-\beta_1, -\beta_2, \cdots, -\beta_N\right] \\ &= Diag\left[-e^{i\omega_j}\beta, -e^{i\omega_j 2}\beta, ..., -e^{i\omega_j N}\beta\right] \\ & \quad Diag\left[-e^{-i\omega_j[N-1]}\beta, -e^{-i\omega_j[N-2]}\beta, ..., \beta\right] \\ \phi_{0(h,k)}^{-22} &= \begin{cases} 1 & h = k,\ k = 1, ..., N \\ -e^{-i\omega_j} & h = k+1,\ k = 1, ..., N \\ 0 & \text{otherwise} \end{cases} \\ \phi_{1(h,k)}^{-22} &= \begin{cases} e^{-i\omega_j} & h = 1,\ k = N \\ 0 & \text{otherwise} \end{cases}, \end{aligned}$$

β is the zero frequency cointegration coefficient of (8) and the subscripts (h, j) refer to the $(h, j)^{th}$ element of the respective sub-matrix, Φ_0^{-12} or Φ_1^{-22}. Note that the inverse of matrix Φ_0^- using results for the inverse of partitioned matrices will be as follows:

$$\left(\Phi_0^-\right)^{-1} = \begin{bmatrix} I_{N\times N} & -\Phi_0^{-12}\left(\Phi_0^{-22}\right)^{-1} \\ 0_{N\times N} & \left(\Phi_0^{-22}\right)^{-1} \end{bmatrix}$$

Note also that sub matrix $\left(\Phi_0^{-22}\right)^{-1}$ as previously stated is the inverse of a $N \times N$ lower-triangular Toeplitz matrix associated to the polynomial $\left(1 - e^{-i\omega_j}L\right)$. Hence matrix $\left(\Phi_0^-\right)^{-1}$ will collect the coefficients on the expansion of the inverse polynomial associated to $\left(1 - e^{-i\omega_j}L\right)$. Once we know the form of $\left(\Phi_0^-\right)^{-1}$ it is easy to check that the resulting matrix $\left(\Phi_0^-\right)^{-1}\Phi_1^-$ will be a $2N \times 2N$ matrix with its first columns with all the elements equal to zero and the last column collecting in first place the elements of the first column of $-\Phi_0^{-12}\left(\Phi_0^{-22}\right)^{-1}$ multiplied by[8] $e^{-i\omega_j}$, followed by the first column of matrix $\left(\Phi_0^{-22}\right)^{-1}$ multiplied by $e^{-i\omega_j}$ as well[9], hence as the last element of the last column of $\left(\Phi_0^-\right)^{-1}\Phi_1^-$ is equal to $e^{-iN\omega_j} = 1$. it is clear that matrix $\left(\Phi_0^-\right)^{-1}\Phi_1^-$ is idempotent.

- 1. Recursive substitution from (35), yields

$$Z_\tau^{(0,\omega_j)-} = \left(\Phi_0^-\right)^{-1}\Phi_1^- Z_0^{(0,\omega_j)-} + \left(\Phi_0^-\right)^{-1} V_\tau^Z + \left(\Phi_0^-\right)^{-1}\Phi_1^-\left(\Phi_0^-\right)^{-1}\sum_{j=1}^{\tau-1} V_{\tau-j}^Z \tag{36}$$

We can write

$$\left(\Phi_0^-\right)^{-1}\Phi_1^-\left(\Phi_0^-\right)^{-1} = \begin{bmatrix} 0_{N\times N} & \mathbf{C}_y^{(0)-} \\ 0_{N\times N} & \mathbf{C}_x^- \end{bmatrix} \tag{37}$$

where the $N \times N$ sub-matrices $\mathbf{C}_y^{(0)-}$ and $\mathbf{C}_x^-$ each have rank one, so that

$$\begin{aligned} \mathbf{C}_x^- &= a_x^- b_x^{-\prime} \\ \mathbf{C}_y^{(0)-} &= a_y^{(0)-} b_x^{-\prime} \end{aligned} \tag{38}$$

with

$$\begin{aligned} a_x^- &= \begin{bmatrix} 1 & e^{-i\omega_j} & e^{-i2\omega_j} & \cdots & e^{-i(N-2)\omega_j} & e^{-i(N-1)\omega_j} \end{bmatrix}' \\ a_y^{(0)-} &= e^{i\omega_j}\beta\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix}' \\ b_x^- &= b_y^- = \begin{bmatrix} 1 & e^{-i(N-1)\omega_j} & e^{-i(N-2)\omega_j} & \cdots & e^{-i2\omega_j} & e^{-i\omega_j} \end{bmatrix}'. \end{aligned} \tag{39}$$

[8] That is $e^{-i\omega_j}\left[\beta_1, e^{-i\omega_j}\beta_2, \cdots, e^{-i\omega_j[N-1]}\beta_N\right]' = e^{-i\omega_j}\left[e^{-i\omega_j[N-1]}\beta, e^{-i\omega_j[N-2]}e^{-i\omega_j}\beta, \cdots, e^{-i\omega_j[N-1]}\beta\right]' = \beta[1, 1, \cdots, 1]$.

[9] That is $e^{-i\omega_j}\left[1, e^{-i\omega_j}, \cdots, e^{-i\omega_j[N-1]}\right]' = \left[e^{-i\omega_j}, e^{-i2\omega_j}, \cdots, e^{-iN\omega_j}\right]'$

It is clear from (36) to (39) that $\left(\Phi_0^-\right)^{-1}\Phi_1^-\left(\Phi_0^-\right)^{-1}$, $\mathbf{C}_y^{(0)-}$ and $\mathbf{C}_x^-$ have rank 1, and hence there is one common stochastic trend shared by the seasons of both time series $y_{n\tau}^{(0)-}$ and $x_{n\tau}^-$ on equivalently we have $2N-1$ cointegration relationships between the seasons of the vector Z_τ^-. Note also that $\mathbf{C}_x^-$ in (38) has the same expression that (33) them it is possible to write:

$$\begin{aligned}\mathbf{C}_x^- &= (\mathbf{a}_x^-)(\mathbf{b}_x^-)' = (\mathbf{a}_x^-)e^{-i\omega_j}e^{i\omega_j}(\mathbf{b}_x^-)' = \mathbf{v}_x^-\mathbf{v}_x^{+\prime} \\ \mathbf{v}_x^- &= \left[\begin{array}{ccccc} e^{-i\omega_j} & e^{-i2\omega_j} & e^{-i3\omega_j} & \cdots & e^{-iN\omega_j}\end{array}\right]' \\ \mathbf{v}_x^+ &= \left[\begin{array}{ccccc} e^{i\omega_j} & e^{i2\omega_j} & e^{i3\omega_j} & \cdots & e^{iN\omega_j}\end{array}\right]'.\end{aligned} \tag{40}$$

And in the case of $\mathbf{C}_y^{(0)-}$ note that:

$$\mathbf{C}_y^{(0)-} = a_y^{(0)-}b_x^{-\prime} = \beta\mathbf{1}e^{i\omega_j}b_x^{-\prime} = \beta\mathbf{1}\mathbf{v}_x^{+\prime} \tag{41}$$

with $\mathbf{1}$ a $N\times 1$ vector of ones. Based of the fact that:

$$\frac{1}{\sqrt{T}}\sum_{j=1}^{\lfloor Tr\rfloor}V_\tau^Z = \left[\begin{array}{c}\frac{1}{\sqrt{T}}\sum_{j=1}^{\lfloor Tr\rfloor}V_\tau^u \\ \frac{1}{\sqrt{T}}\sum_{j=1}^{\lfloor Tr\rfloor}V_\tau^u\end{array}\right] \Rightarrow \left[\begin{array}{c}\sigma_u W^u(r) \\ \sigma W^v(r)\end{array}\right]$$

where $W^u(r)$ and $W^v(r)$ are two $N\times 1$ vector Brownian motions. It is possible to see that for Z_τ^- in (36) we have:

$$\frac{1}{\sqrt{T}}Z_{\lfloor Tr\rfloor}^{(0,\omega_j)-} \Rightarrow \left[\begin{array}{cc}0_{N\times N} & \mathbf{C}_y^{(0)-} \\ 0_{N\times N} & \mathbf{C}_x^-\end{array}\right]\left[\begin{array}{c}\sigma_u W^u(r) \\ \sigma W^v(r)\end{array}\right] \tag{42}$$

and the result in Lemma 5 came from taking together (42),(40) and (41).■

Lemma 6 *For $Z_\tau^{(\omega_k,\omega_j)-} = \left[y_{1\tau}^-, y_{2\tau}^-, \ldots, y_{N\tau}^-, x_{1\tau}^-, x_{2\tau}^-, \ldots, x_{N\tau}^-\right]'$ with $x_{n\tau}^-$ and $y_{n\tau}^- n = 1, 2, \cdots, N$ defined in (11) and with $v_{n\tau}\sim iid\left(0,\sigma^2\right)$ and $u_{n\tau}\sim iid\left(0,\sigma_u^2\right)$ it is possible to write:*

$$\frac{1}{\sqrt{T}}Z_{\lfloor Tr\rfloor}^{(\omega_k,\omega_j)-} \Rightarrow \sigma\left(N/2\right)^{1/2}\left[\begin{array}{c}\beta\mathbf{v}_k^- \\ \mathbf{v}_j^-\end{array}\right]\left(w_R^v(r) + iw_I^v(r)\right) \tag{43}$$

with $\left(w_R^v(r) + iw_I^v(r)\right)$ and $\mathbf{v}_j^-$ as in Lemma 1 and finally $\mathbf{v}_k^- = \left[\begin{array}{ccccc} e^{-i\omega_k} & e^{-i2\omega_k} & e^{-i3\omega_k} & \cdots & e^{-iN\omega_k}\end{array}\right]'$.

Remark 7 *As particular case of (11) and (43) we could define a triangular system between a couple of complex valued integrated processes one associated to the Nyquist frequency (π) and another associated to an harmonic frequency ω_j, if we multiply (10) by $e^{-i\pi(N(\tau-1)+n)}$.*

Proof of Lemma 6: In the case of triangular system (11) we then define the vectors $Z_\tau^{(\omega_k,\omega_j)-} = \left[y_{1\tau}^-, y_{2\tau}^-, \cdots, y_{N\tau}^-, x_{1\tau}^-, x_{2\tau}^-, \cdots, x_{N\tau}^-\right]'$ and $V_\tau^Z = [u_{1\tau}, u_{2\tau}, \cdots, u_{N\tau}, v_{1\tau}, v_{2\tau}, \cdots, v_{N\tau}]' = [V_\tau^{u\prime}V_\tau']'$ with $V_\tau = [v_{1\tau}, v_{2\tau}, \cdots, v_{N\tau}]'$ as in Lemma 1 and $V_\tau^u = \left[-e^{i\omega_k}u_{1\tau}, -e^{iw\omega_k}u_{2\tau}, \cdots, -e^{iN\omega_k}u_{N\tau}\right]'$, using the same line of argument as in the proof of Lemma 5 we can write

$$\Phi_0^- Z_\tau^{(\omega_k,\omega_j)-} = \Phi_1^- Z_{\tau-1}^{(\omega_k,\omega_j)-} + V_\tau^Z \tag{44}$$

where

$$\Phi_0^- = \left[\begin{array}{cc} I_{N\times N} & \Phi_0^{-12} \\ 0_{N\times N} & \Phi_0^{-22}\end{array}\right], \quad \Phi_1^- = \left[\begin{array}{cc} 0_{N\times N} & 0_{N\times N} \\ 0_{N\times N} & \Phi_1^{-22}\end{array}\right]$$

in which all sub-matrix are $N\times N$, with

$$\begin{aligned}\Phi_0^{-12} &= Diag\left[-\beta_1, -\beta_2, \cdots, -\beta_N\right] \\ &= Diag\left[-e^{i(\omega_j-\omega_k)}\beta, -e^{i(\omega_j-\omega_k)2}\beta, ..., -e^{i(\omega_j-\omega_k)N}\beta\right] \\ & Diag\left[-e^{-i(\omega_j-\omega_k)[N-1]}\beta, -e^{-i(\omega_j-\omega_k)[N-1]}\beta, ..., -e^{-i(\omega_j-\omega_k)}\beta\right] \\ \phi_{0(h,k)}^{-22} &= \begin{cases} 1 & h=k,\ k=1,...,N \\ -e^{-i\omega_j} & h=k+1,\ k=1,...,N \\ 0 & \text{otherwise}\end{cases} \\ \phi_{1(h,k)}^{-22} &= \begin{cases} e^{-i\omega_j} & h=1,\ k=N \\ 0 & \text{otherwise}\end{cases},\end{aligned}$$

Recursive substitution from (44), and recognizing that $\left(\Phi_0^-\right)^{-1}\Phi_1^-$ is idempotent[10], yields

$$Z_\tau^{(\omega_k,\omega_j)-} = \left(\Phi_0^-\right)^{-1}\Phi_1^- Z_0^{(\omega_k,\omega_j)-} + \left(\Phi_0^-\right)^{-1} V_\tau^Z + \left(\Phi_0^-\right)^{-1}\Phi_1^- \left(\Phi_0^-\right)^{-1}\sum_{j=1}^{\tau-1} V_{\tau-j}^Z \tag{45}$$

We can write

$$\left(\Phi_0^-\right)^{-1}\Phi_1^-\left(\Phi_0^-\right)^{-1} = \begin{bmatrix} 0_{N\times N} & \mathbf{C}_y^- \\ 0_{N\times N} & \mathbf{C}_x^- \end{bmatrix} \tag{46}$$

where the $N \times N$ sub-matrices $\mathbf{C}_y$ and $\mathbf{C}_x^-$ each have rank one, so that

$$\begin{aligned} \mathbf{C}_x^- &= a_x^- b_x^{-\prime} \\ \mathbf{C}_y^- &= a_y^- b_x^{-\prime} \end{aligned} \tag{47}$$

with

$$\begin{aligned} a_x^- &= \begin{bmatrix} 1 & e^{-i\omega_j} & e^{-i2\omega_j} & \cdots & e^{-i(N-2)\omega_j} & e^{-i(N-1)\omega_j} \end{bmatrix}' \\ a_y^- &= e^{-i\omega_k}e^{i\omega_j}\beta \begin{bmatrix} 1 & e^{-i\omega_k} & e^{-i2\omega_k} & \cdots & e^{-i(N-2)\omega_k} & e^{-i(N-1)\omega_k} \end{bmatrix}' \\ b_x^- &= b_y^- = \begin{bmatrix} 1 & e^{-i(N-1)\omega_j} & e^{-i(N-2)\omega_j} & \cdots & e^{-i2\omega_j} & e^{-i\omega_j} \end{bmatrix}'. \end{aligned} \tag{48}$$

It is clear from (46) to (47) that $\left(\Phi_0^-\right)^{-1}\Phi_1^-\left(\Phi_0^-\right)^{-1}$, $\mathbf{C}_y^-$ and $\mathbf{C}_x^-$ have rank 1, and hence there is one common stochastic trend shared by the seasons of both time series $y_{n\tau}^-$and $x_{n\tau}^-$ on equivalently we have $2N-1$ cointegration relationships between the seasons of the vector $Z_\tau^{(\omega_k,\omega_j)-}$. Note also that $\mathbf{C}_x^-$ in (47) has the same expression that (33) them it is possible to write:

$$\begin{aligned} \mathbf{C}_x^- &= \mathbf{v}_x^-\mathbf{v}_x^{+\prime} \\ \mathbf{v}_x^- &= \begin{bmatrix} e^{-i\omega_j} & e^{-i2\omega_j} & e^{-i3\omega_j} & \cdots & e^{-iN\omega_j} \end{bmatrix}' \\ \mathbf{v}_x^{-+} &= \begin{bmatrix} e^{i\omega_j} & e^{i2\omega_j} & e^{i3\omega_j} & \cdots & e^{iN\omega_j} \end{bmatrix}'. \end{aligned} \tag{49}$$

And in the case of $\mathbf{C}_y^-$ note that:

$$\begin{aligned} \mathbf{C}_y^- &= a_y^- b_x^{-\prime} = \beta\mathbf{v}_y^-\mathbf{v}_x^{+\prime} \\ \mathbf{v}_y^- &= \begin{bmatrix} e^{-i\omega_k} & e^{-i2\omega_k} & e^{-i3\omega_k} & \cdots & e^{-iN\omega_k} \end{bmatrix}'. \end{aligned} \tag{50}$$

Based of the fact that:

$$\frac{1}{\sqrt{T}}\sum_{j=1}^{\lfloor Tr\rfloor} V_\tau^Z = \begin{bmatrix} \frac{1}{\sqrt{T}}\sum_{j=1}^{\lfloor Tr\rfloor} V_\tau^u \\ \frac{1}{\sqrt{T}}\sum_{j=1}^{\lfloor Tr\rfloor} V_\tau^u \end{bmatrix} \Rightarrow \begin{bmatrix} \sigma_u W^u(r) \\ \sigma W^v(r) \end{bmatrix}$$

where $W^v(r)$ is a $N \times 1$ vector Brownian motion as in Lemma 2 and $W^u(r)$ is a $N \times 1$ vector complex valued Brownian motion. Hence it is possible to see that for $Z_\tau^{(\omega_k,\omega_j)-}$ in (45) we have:

$$\frac{1}{\sqrt{T}} Z_{\lfloor Tr\rfloor}^{(0,\omega_j)-} \Rightarrow \begin{bmatrix} 0_{N\times N} & \mathbf{C}_y^- \\ 0_{N\times N} & \mathbf{C}_x^- \end{bmatrix} \begin{bmatrix} \sigma_u W^u(r) \\ \sigma W^v(r) \end{bmatrix} \tag{51}$$

and the result in Lemma 3 came from taking together (51),(49) and (50).■

[10]First note that the for the inverse of Φ_0^- we have:

$$\left(\Phi_0^-\right)^{-1} = \begin{bmatrix} I_{N\times N} & -\Phi_0^{-12}\left(\Phi_0^{-22}\right)^{-1} \\ 0_{N\times N} & \left(\Phi_0^{-22}\right)^{-1} \end{bmatrix}.$$

Note also that the resulting matrix $\left(\Phi_0^-\right)^{-1}\Phi_1^-$ is a $2N \times 2N$ matrix with its first colunms with all the elements equal to zero and the last column collecting in first place the elements of the first column of $-\Phi_0^{-12}\left(\Phi_0^{-22}\right)^{-1}$ multiplied by $e^{-i\omega_j}$, followed by the first column of matrix $\left(\Phi_0^{-22}\right)^{-1}$ multiplied by $e^{-i\omega_j}$ as well, in this case also the last element of the last column of $\left(\Phi_0^-\right)^{-1}\Phi_1^-$ is equal to $e^{-iN\omega_j} = 1$. it is clear that matrix $\left(\Phi_0^-\right)^{-1}\Phi_1^-$ is idempotent.

Lemma 8 *For* $Z_\tau^{(0,\omega_j)} = \left[\left(y_{1\tau}^{(0)-}+y_{1\tau}^{(0)+}\right)/2, \left(y_{2\tau}^{(0)-}+y_{2\tau}^{(0)+}\right)/2, \ldots, \left(y_{N\tau}^{(0)-}+y_{N\tau}^{(0)+}\right)/2, x_{1\tau}, x_{2\tau}, \ldots, x_{N\tau}\right]'$ *with* $x_{n\tau}$ *and* $\left(y_{n\tau}^{(0)-}+y_{n\tau}^{(0)+}\right)/2$ $n = 1, 2, \cdots, N$ *defined in (16) and with* $v_{n\tau} \sim iid\left(0,\sigma^2\right)$*,*$u_{n\tau} \sim iid\left(0,\sigma_u^2\right)$ *and* $u_{n\tau} \sim iid\left(0,\sigma_u^2\right)$ *it is possible to write:*

$$\frac{1}{\sqrt{T}} Z_{\lfloor Tr\rfloor}^{(0,\omega_j)} \Rightarrow \sigma\left(N/2\right)^{1/2} \left[\begin{array}{c} \frac{1}{2}\left[\left(\beta_R+i\beta_I\right)\mathbf{1}\left(w_R^v\left(r\right)+iw_I^v\left(r\right)\right)+\left(\beta_R-i\beta_I\right)\mathbf{1}\left(w_R^v\left(r\right)-iw_I^v\left(r\right)\right)\right] \\ \frac{e^{-i\omega_j}}{-2i\sin(\omega_j)}\mathbf{v}_j^-\left(w_R^v\left(r\right)+iw_I^v\left(r\right)\right)+\frac{e^{-i\omega_j}}{2i\sin(\omega_j)}\mathbf{v}_j^-\left(w_R^v\left(r\right)-iw_I^v\left(r\right)\right) \end{array}\right] \quad (52)$$

with $\left(w_R^v\left(r\right)+iw_I^v\left(r\right)\right)$*,* $\mathbf{v}_j^-$ *and* $\mathbf{v}_j^+$ *as in Lemma 1,* $\left(w_R^v\left(r\right)-iw_I^v\left(r\right)\right)$ *the complex conjugate of* $\left(w_R^v\left(r\right)+iw_I^v\left(r\right)\right)$ *and* $\mathbf{1}$ *and* $N\times 1$ *vector of ones.*

Proof of Lemma 8: In the case of triangular system (16) we then define the vectors $Z_\tau^{(0,\omega_j)} = \left[\left(y_{1\tau}^{(0)-}+y_{1\tau}^{(0)+}\right)/2, \left(y_{2\tau}^{(0)-}+y_{2\tau}^{(0)+}\right)/2, \ldots, \left(y_{N\tau}^{(0)-}+y_{N\tau}^{(0)+}\right)/2, x_{1\tau}, x_{2\tau}, \ldots, x'_{N\tau}\right]$ and $V_\tau^Z = [\left(u_{1\tau}+\bar{u}_{1\tau}\right)/2, \left(u_{2\tau}+\bar{u}_{2\tau}\right)/2, \left(u_{N\tau}+\bar{u}_{N\tau}\right)/2, v_{1\tau}, v_{2\tau}, \cdots, v'_{N\tau}] = [V_\tau^{u\prime}V_\tau']'$ with $V_\tau = [v_{1\tau}, v_{2\tau}, \cdots, v_{N\tau}]'$ and $V_\tau^u = [\left(u_{1\tau}+\bar{u}_{1\tau}\right)/2, \left(u_{2\tau}+\bar{u}_{2\tau}\right)/2, \cdots, \left(u_{N\tau}+\bar{u}_{N\tau}\right)/2]'$, using the same line of argument as in the proof of previous lemmas we can write

$$\Phi_0 Z_\tau^{(0,\omega_j)} = \Phi_1 Z_\tau^{(0,\omega_j)} + V_\tau^Z \quad (53)$$

where

$$\Phi_0 = \left[\begin{array}{cc} I_{N\times N} & \Phi_0^{12} \\ 0_{N\times N} & \Phi_0^{22} \end{array}\right], \quad \Phi_1 = \left[\begin{array}{cc} 0_{N\times N} & \Phi_1^{12} \\ 0_{N\times N} & \Phi_1^{22} \end{array}\right]$$

in which all sub-matrix are $N\times N$, with

$$\Phi_0^{12} = \left[\begin{array}{cccccc} -\beta_{11} & 0 & 0 & 0 & \cdots & 0 \\ -\beta_{22} & -\beta_{12} & 0 & 0 & \cdots & 0 \\ 0 & -\beta_{23} & -\beta_{13} & 0 & \cdots & 0 \\ 0 & 0 & -\beta_{24} & -\beta_{14} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\beta_{2N} & -\beta_{1N} \end{array}\right]$$

with β_{1n} and β_{2n} defined in (17).

$$\Phi_0^{22} = \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & \cdots & 0 \\ -2\cos\left(\omega_j\right) & 1 & 0 & 0 & \cdots & 0 \\ 1 & -2\cos\left(\omega_j\right) & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2\cos\left(\omega_j\right) & 1 & \cdots & 0 \\ 0 & 0 & 1 & -2\cos\left(\omega_j\right) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & -2\cos\left(\omega_j\right) & 1 \end{array}\right]$$

$$\Phi_1^{12} = \left[\begin{array}{cccccc} 0 & 0 & 0 & \cdots & 0 & \beta_{12} \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -0 \end{array}\right]$$

$$\Phi_1^{22} = \left[\begin{array}{cccccc} 0 & 0 & \cdots & 0 & -1 & 2\cos\left(\omega_j\right) \\ 0 & 0 & \cdots & 0 & 0 & -1 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & -0 \end{array}\right]$$

First note that, using results form the inverse of a partitioned matrix, we have:

$$\left(\Phi_0\right)^{-1} = \left[\begin{array}{cc} I_{N\times N} & -\Phi_0^{12}\left(\Phi_0^{22}\right)^{-1} \\ 0_{N\times N} & \left(\Phi_0^{22}\right)^{-1} \end{array}\right].$$

Matrix Φ_0^{22} is a lower-triangular Toeplitz matrix associated to the polynomial $\left(1 - 2\cos(\omega_j) L + L^2\right)$, hence $\left(\Phi_0^{22}\right)^{-1}$ will collect the coefficients of the expansion of the inverse of $\left(1 - 2\cos(\omega_j) L + L^2\right)$, that is:

$$\left(\Phi_0^{22}\right)^{-1} = \frac{1}{\sin(\omega_j)} \begin{bmatrix} \sin(\omega_j) & 0 & 0 & 0 & \cdots & 0 \\ \sin(2\omega_j) & \sin(\omega_j) & 0 & 0 & \cdots & 0 \\ \sin(3\omega_j) & \sin(2\omega_j) & \sin(\omega_j) & 0 & \cdots & 0 \\ \sin(4\omega_j) & \sin(3\omega_j) & \sin(2\omega_j) & \sin(\omega_j) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \sin(N\omega_j) & \sin([N-1]\omega_j) & \sin([N-2]\omega_j) & \sin([N-3]\omega_j) & \cdots & \sin(\omega_j) \end{bmatrix}$$

In the case of the case of $-\Phi_0^{12}\left(\Phi_0^{22}\right)^{-1}$ it is easy to check that:

$$\begin{aligned} -\Phi_0^{12}\left(\Phi_0^{22}\right)^{-1} &= \frac{1}{\sin(\omega_j)} \times \\ & \begin{bmatrix} \beta_{11}\sin(\omega_j) & 0 & \cdots & 0 \\ \beta_{12}\sin(2\omega_j) + \beta_{22}\sin(\omega_j) & \beta_{12}\sin(\omega_j) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{1N}\sin(N\omega_j) + \beta_{2N}\sin([N-1]\omega_j) & \beta_{1N}\sin([N-1]\omega_j) + \beta_{2N}\sin([N-2]\omega_j) & \cdots & \beta_{1N}\sin(\omega_j \end{bmatrix} \end{aligned}$$

Matrix $(\Phi_0)^{-1}\Phi_1$ is a $2N \times 2N$ matrix, but as matrix Φ_1 only has 4 elements different from zero (see de definition Φ_1^{12} and Φ_1^{22} above) it is possible to check that for $(\Phi_0)^{-1}\Phi_1$ it is possible to write:

$$(\Phi_0)^{-1}\Phi_1 = \begin{bmatrix} 0_{2N\times 2(N-1)} & \mathbf{V}_y \\ 0_{2N\times 2(N-1)} & \mathbf{V}_x \end{bmatrix} \tag{54}$$

where $\mathbf{V}_y$ and $\mathbf{V}_x$ are $S \times 2$ matrices with the following form[11]:

$$\mathbf{V}_y = \frac{1}{\sin(\omega_j)} \begin{bmatrix} -[\beta_{11}\sin(\omega_j)] & \beta_{11}\sin(2\omega_j) + \beta_{12}\sin(\omega_j) \\ -[\beta_{12}\sin(2\omega_j) + \beta_{22}\sin(\omega_j)] & \beta_{12}\sin(3\omega_j) + \beta_{22}\sin(2\omega_j) \\ -[\beta_{13}\sin(3\omega_j) + \beta_{23}\sin(2\omega_j)] & \beta_{13}\sin(4\omega_j) + \beta_{23}\sin(3\omega_j) \\ -[\beta_{14}\sin(4\omega_j) + \beta_{24}\sin(3\omega_j)] & \beta_{14}\sin(5\omega_j) + \beta_{24}\sin(4\omega_j) \\ \vdots & \vdots \\ -[\beta_{1N}\sin(N\omega_j) + \beta_{2N}\sin([N-1]\omega_j)] & \beta_{1N}\sin([N+1]\omega_j) + \beta_{2N}\sin(N\omega_j) \end{bmatrix} \tag{55}$$

$$\mathbf{V}_x = \frac{1}{\sin(\omega_j)} \begin{bmatrix} -\sin(\omega_j) & \sin(2\omega_j) \\ -\sin(2\omega_j) & \sin(3\omega_j) \\ -\sin(3\omega_j) & \sin(4\omega_j) \\ -\sin(4\omega_j) & \sin(5\omega_j) \\ \vdots & \vdots \\ -\sin(N\omega_j) & \sin([N+1]\omega_j) \end{bmatrix}. \tag{56}$$

It is easy to check that the matrix $\Phi_0^{-1}\Phi_1$ is idempotent[12]. Recursive substitution from (53) yields:

$$Z_\tau^{(0,\omega_j)} = \Phi_0^{-1}\Phi_1^{-} Z_o^{(0,\omega_j)} + \Phi_0^{-1} V_\tau^Z + \Phi_0^{-1}\Phi_1\Phi_0^{-1}\sum_{j=1}^{\tau-1} V_{\tau-j}^Z$$

[11] Based on the form of Φ_1 it is possible to check that the fist colunm of $\mathbf{V}_y$ and $\mathbf{V}_x$ is going to be equal to minus th first column of $\left(\Phi_0^{22}\right)^{-1}$ and $-\Phi_0^{12}\left(\Phi_0^{22}\right)^{-1}$ respectively. Finally in the case of the second column of $\mathbf{V}_y$ and $\mathbf{V}_x$ is the weigthed sum of the first and second column of $\left(\Phi_0^{22}\right)^{-1}$ and $-\Phi_0^{12}\left(\Phi_0^{22}\right)^{-1}$ with weigths $2\cos(\omega_k)$ and -1 respectively. Finally note to obtain the second column for (55) and (56) we use the following recurrent expression for multiple angle:

$$\begin{aligned} \sin(j\omega_k) &= 2\cos(\omega_k)\sin([j-1]\omega_k) - \sin([j-2]\omega_k) \\ \cos(j\omega_k) &= 2\cos(\omega_k)\cos([j-1]\omega_k) - \cos([j-2]\omega_k) \end{aligned}$$

[12] The lower 2×2 submatrix of $\mathbf{V}_x$ is equal to:

$$\begin{bmatrix} \frac{-\sin([N-1]\omega_j)}{\sin(\omega_j)} & \frac{\sin(N\omega_j)}{\sin(\omega_j)} \\ \frac{-\sin(N\omega_j)}{\sin(\omega_j)} & \frac{\sin([N+1]\omega_k)}{\sin(\omega_j)} \end{bmatrix}$$

and it is equal to $I_{2\times 2}$. Hence $\left[(\Phi_0)^{-1}\Phi_1\right]^h = (\Phi_0)^{-1}\Phi_1$ for $h = 2, 3, \ldots$.

and matrix $\Phi_0^{-1}\Phi_1\Phi_0^{-1}$ displays the impact of the accumulation shocks $\sum_{j=1}^{\tau-1} V_{\tau-j}^{Z}$ and has the form:

$$\begin{aligned}\Phi_0^{-1}\Phi_1\Phi_0^{-1} &= \begin{bmatrix} 0_{N\times N} & \mathbf{V}_y\mathbf{U}_x' \\ 0_{N\times N} & \mathbf{V}_x\mathbf{U}_x' \end{bmatrix} \\ \mathbf{U}_x' &= \frac{1}{\sin(\omega_j)}\begin{bmatrix} \sin([N-1]\,\omega_j) & \sin([N-2]\,\omega_j) & \sin([N-3]\,\omega_j) & \sin([N-4]\,\omega_j) & \cdots & 0 \\ \sin(N\omega_j) & \sin([N-1]\,\omega_j) & \sin([N-2]\,\omega_j) & \sin([N-3]\,\omega_j) & \cdots & \sin(\omega_j) \end{bmatrix}.\end{aligned}$$

First we are going to pay attention to $\mathbf{V}_x\mathbf{U}_x'$, in this case it is check that:

$$\begin{aligned}\mathbf{V}_x\mathbf{U}_x' &= \mathbf{C}_j = \mathrm{Ci}\,rc\left[\frac{\sin(\omega_j)}{\sin(\omega_j)}, \frac{\sin(N\omega_j)}{\sin(\omega_j)}, \cdots, \frac{\sin(2\omega_j)}{\sin(\omega_j)}\right] \\ &= \frac{e^{-i\omega_j}}{-2i\sin(\omega_j)}\mathbf{C}_j^{-} + \frac{e^{i\omega_j}}{2i\sin(\omega_j)}\mathbf{C}_j^{+}. \end{aligned} \tag{57}$$

Where $\mathbf{C}_j$ is defined in remark 2 and $\mathbf{C}_j^{-}$and $\mathbf{C}_j^{+}$ in Lemma 1. In the case of $\mathbf{V}_y$ it is possible to show that (55) could be simplified as follows:

$$\mathbf{V}_y = \begin{bmatrix} -[\beta_R\cos(\omega_j) - \beta_I\sin(\omega_j)] & \beta_R \\ -[\beta_R\cos(\omega_j) - \beta_I\sin(\omega_j)] & \beta_R \\ -[\beta_R\cos(\omega_j) - \beta_I\sin(\omega_j)] & \beta_R \\ -[\beta_R\cos(\omega_j) - \beta_I\sin(\omega_j)] & \beta_R \\ \vdots & \vdots \\ -[\beta_R\cos(\omega_j) - \beta_I\sin(\omega_j)] & \beta_R \end{bmatrix} \tag{58}$$

In order to obtain (58) form (55), we use the expression of β_{1n} and β_{2n} in (17) and the formula connecting products of sin and cos to sums[13] jointly with the expressions for the double angle sin and cos[14]. Clearly the two columns of $\mathbf{V}_y$ are linearly dependent. Note also that it is possible to check that $\mathbf{V}_y\mathbf{U}_x'$ is a matrix with generic row element $\beta_R\cos(\omega_j h) - \beta_I\sin(\omega_j h)$ with $h = 1, 2, \ldots, N$, that is, all the element of the first column are equal to $\beta_R\cos(\omega_j) - \beta_I\sin(\omega_j)$, the elements of the second column to $\beta_R\cos(\omega_j 2) - \beta_I\sin(\omega_j 2)$, and so on a so forth. Finally it is possible to check that for $\mathbf{V}_y\mathbf{U}_x'$ it is possible to write:

$$\mathbf{V}_y\mathbf{U}_x' = \frac{1}{2}\left[(\beta_R + i\beta_I)\,\mathbf{1}\mathbf{v}_j^{+\prime} + (\beta_R - i\beta_I)\,\mathbf{1}\mathbf{v}_j^{-\prime}\right].$$

Where $\mathbf{1}$ is a $N\times 1$ vector of ones and $\mathbf{v}_j^{-} = \begin{bmatrix} e^{-i\omega_j} & e^{-i2\omega_j} & e^{-i3\omega_j} & \cdots & e^{-iN\omega_j} \end{bmatrix}'$ $\mathbf{v}_j^{+} = \begin{bmatrix} e^{i\omega_j} & e^{i2\omega_j} & e^{i3\omega_j} & \cdots & e^{iN\omega_j} \end{bmatrix}'$. Hence the result in (52) it is easily obtained, following the lines of the proofs of the previous lemmas.■

Lemma 9 *For $Z_\tau^{(\omega_k,\omega_j)} = [y_{1\tau}, y_{2\tau}, \ldots, y_{N\tau}, x_{1\tau}, x_{2\tau}, \ldots, x_{N\tau}]'$ with $x_{n\tau}$ and $y_{1\tau}$ $n = 1, 2, \cdots, N$ defined in (24) and with $v_{n\tau} \sim iid\left(0, \sigma^2\right)$ and $\varpi_{n\tau} \sim iid\left(0, \sigma_\varpi^2\right)$ it is possible to write:*

$$\frac{1}{\sqrt{T}}Z_{\lfloor T r\rfloor}^{(\omega_k,\omega_j)} \Rightarrow \sigma(N/2)^{1/2}\begin{bmatrix} \left\{\frac{(\beta_R+i\beta_I)}{2}\mathbf{v}_k^{-}\left(w_R^v(r) + iw_I^v(r)\right) + \frac{(\beta_R+i\beta_I)}{2}\mathbf{v}_k^{+}\left(w_R^v(r) - iw_I^v(r)\right) + \right. \\ \left. +\frac{\cos(\omega_k)}{\sin(\omega_k)}\left[\frac{(\beta_R+i\beta_I)}{-2i}\mathbf{v}_k^{-}\left(w_R^v(r) + iw_I^v(r)\right) + \frac{(\beta_R+i\beta_I)}{2i}\mathbf{v}_k^{+}\left(w_R^v(r) - iw_I^v(r)\right)\right]\right\} \\ \left\{\frac{e^{-i\omega_j}}{-2i\sin(\omega_j)}\mathbf{v}_j^{-}\left(w_R^v(r) + iw_I^v(r)\right) + \frac{e^{-i\omega_j}}{2i\sin(\omega_j)}\mathbf{v}_j^{-}\left(w_R^v(r) - iw_I^v(r)\right)\right\} \end{bmatrix} \tag{59}$$

with $\left(w_R^v(r) \pm iw_I^v(r)\right)$, $\mathbf{v}_j^{-}$ and $\mathbf{v}_j^{+}$ as in Lemma 8, and $\mathbf{v}_k^{-} = \begin{bmatrix} e^{-i\omega_k} & e^{-i2\omega_k} & e^{-i3\omega_k} & \cdots & e^{-iN\omega_k} \end{bmatrix}'$ and $\mathbf{v}_k^{+} = \begin{bmatrix} e^{i\omega_k} & e^{i2\omega_k} & e^{i3\omega_k} & \cdots & e^{iN\omega_k} \end{bmatrix}'$.

Proof of Lemma 9: In the case of triangular system (24) we then define the vectors $Z_\tau^{(\omega_k,\omega_j)} = [y_{1\tau}, y_{2\tau}, \ldots, y_{N\tau}, x_{1\tau}, x_{2\tau}, \ldots, x_{N\tau}]'$ and $V_\tau^Z = [\varpi_{1\tau}, \varpi_{2\tau}, \cdots, \varpi_{N\tau}, v_{1\tau}, v_{2\tau}, \cdots, v_{N\tau}]' = [V_\tau^{\varpi\prime} V_\tau']'$ with $V_\tau = [v_{1\tau}, v_{2\tau}, \cdots, v_{N\tau}]'$ and $V_\tau^{\varpi} = [\varpi_{1\tau}, \varpi_{2\tau}, \cdots, \varpi_{N\tau}]'$, using the same line of argument as in the proof of previous lemmas we can write

$$\Phi_0 Z_\tau^{(\omega_k,\omega_j)} = \Phi_1 Z_\tau^{(\omega_k,\omega_j)} + V_\tau^Z \tag{60}$$

[13] $\cos(\omega)\sin(\varphi) = \frac{\sin(\omega+\varphi)-\sin(\omega-\varphi)}{2}$ and $\sin(\omega)\sin(\varphi) = \frac{\cos(\omega-\varphi)-\cos(\omega+\varphi)}{2}$

[14] $\sin(2\omega) = 2\cos(\omega)\sin(\omega)$ and $\frac{1-\cos(2\omega)}{2\sin(\omega)} = \sin(\omega)$

where Φ_0 and Φ_1 in (60) have the same expression as in Lemma 8 but in this case β_{1n} and β_{2n} are defined as:

$$\begin{aligned}\beta_{1n} &= \left[\beta_R\left(\cos\left([\omega_k-\omega_j]\,n\right)+\frac{\cos(\omega_k)}{\sin(\omega_k)}\sin\left([\omega_k-\omega_j]\,n\right)\right)\right.\\ &\left.-\beta_I\left(\sin\left([\omega_k-\omega_j]\,n\right)-\frac{\cos(\omega_k)}{\sin(\omega_k)}\cos\left([\omega_k-\omega_j]\,n\right)\right)\right]\\ \beta_{2n} &= -\left[\beta_R\left(\cos\left(\omega_k n-\omega_j\,[n+1]\right)+\frac{\cos(\omega_k)}{\sin(\omega_k)}\sin\left(\omega_k n-\omega_j\,[n+1]\right)\right)\right.\\ &\left.-\beta_I\left(\sin\left(\omega_k n-\omega_j\,[n+1]\right)-\frac{\cos(\omega_k)}{\sin(\omega_k)}\cos\left(\omega_k n-\omega_j\,[n+1]\right)\right)\right]\end{aligned} \tag{61}$$

Hence Φ_0^{12} and Φ_1^{12} are defined as in Lemma 8 but with β_{1n} and β_{2n} following the expression collected in (61), and Φ_0^{22} and Φ_1^{22} with exactly the same expression as in Lemma 8. So also $(\Phi_0)^{-1}$ and $(\Phi_0)^{-1}\Phi_1$ have equivalent expression has in Lemma 8. Clearly $(\Phi_0)^{-1}\Phi_1$ is idempotent and it has the form reported in (54) with $\mathbf{V}_x$ defined in (56) and $\mathbf{V}_y$ as in (55) but with β_{1n} and β_{2n} defined in (61) by recursive substitution from (60) yields:

$$Z_\tau^{(\omega_k,\omega_j)} = \Phi_0^{-1}\Phi_1^{-} Z_o^{(\omega_k,\omega_j)} + \Phi_0^{-1}V_\tau^Z + \Phi_0^{-1}\Phi_1\Phi_0^{-1}\sum_{j=1}^{\tau-1} V_{\tau-j}^Z$$

matrix $\Phi_0^{-1}\Phi_1\Phi_0^{-1}$ is also equal to:

$$\Phi_0^{-1}\Phi_1\Phi_0^{-1} = \begin{bmatrix} 0_{N\times N} & \mathbf{V}_y\mathbf{U}_x' \\ 0_{N\times N} & \mathbf{C}_j \end{bmatrix}$$

$\mathbf{U}_x$ and $\mathbf{C}_j$ are exactly as in Lemma 8 (57) and $\mathbf{V}_y$ is equal to (55) with β_{1n} and β_{2n} defined in (61). Replacing β_{1n} and β_{2n} defined in (61) in (55) and using the expressions $\cos(\omega)\sin(\varphi) = \frac{\sin(\omega+\varphi)-\sin(\omega-\varphi)}{2}$,

$\sin(\omega)\sin(\varphi) = \frac{\cos(\omega-\varphi)-\cos(\omega+\varphi)}{2}$, $\sin(2\omega) = 2\cos(\omega)\sin(\omega)$ and $\frac{1-\cos(2\omega)}{2\sin(\omega)} = \sin(\omega)$ we obtain that the generic element on the first column of $\mathbf{V}_y$ will be:

$$\begin{aligned}&-\beta_R\left[\cos(\omega_k h-\omega_j)+\frac{\cos(\omega_k)}{\sin(\omega_k)}\sin(\omega_k h-\omega_j)\right]-\\ &-\beta_I\left[\cos(\omega_k h-\omega_j)-\frac{\cos(\omega_k)}{\sin(\omega_k)}\sin(\omega_k h-\omega_j)\right]\end{aligned} \tag{62}$$

and the generic element for the second column of $\mathbf{V}_y$ will be:

$$\begin{aligned}&\beta_R\left[\cos(\omega_k h)+\frac{\cos(\omega_k)}{\sin(\omega_k)}\sin(\omega_k h)\right]+\\ &+\beta_I\left[\cos(\omega_k h)-\frac{\cos(\omega_k)}{\sin(\omega_k)}\sin(\omega_k h)\right].\end{aligned} \tag{63}$$

With h in (62) an (63) been the row position of in each column, that is $h = 1,\ldots,N$.

It is possible to check (using trigonometric identities) that the generic element of $\mathbf{V}_y\mathbf{U}_x'$ is going to have the following expression:

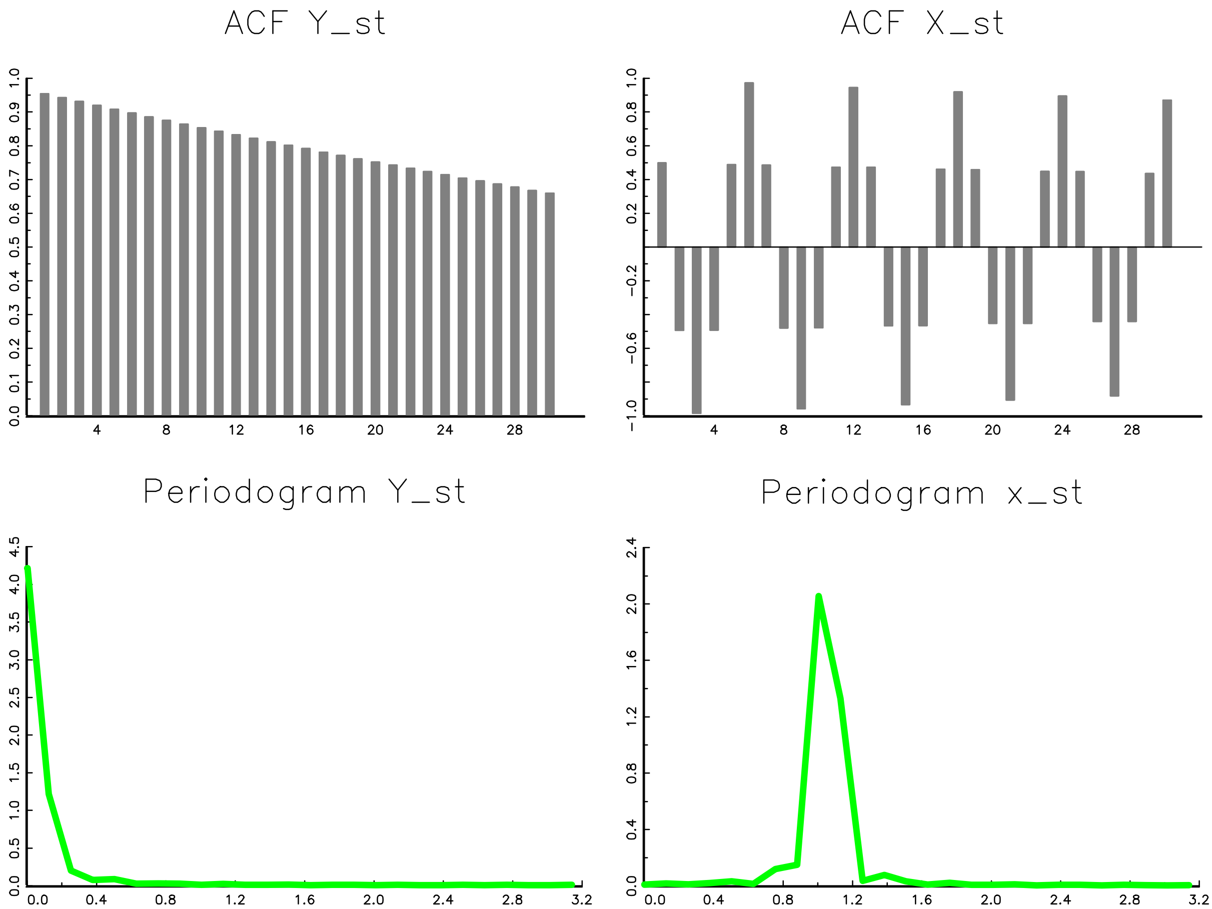
$$\begin{aligned}&\beta_R\left[\cos(\omega_k h-\omega_j f)+\frac{\cos(\omega_k)}{\sin(\omega_k)}\sin(\omega_k h-\omega_j f)\right]+\\ &+\beta_I\left[\sin(\omega_k h-\omega_j f)-\frac{\cos(\omega_k)}{\sin(\omega_k)}\cos(\omega_k h-\omega_j f)\right]\\ h &= 1,\ldots,N\\ j &= 1,\ldots,N\end{aligned} \tag{64}$$

Where h refers to the rows position an f to the column position in $\mathbf{V}_y\mathbf{U}_x'$. Finally it is possible to write:

$$\begin{aligned}\mathbf{V}_y\mathbf{U}_x' &= \left\{\frac{1}{2}\left[(\beta_R+i\beta_I)\,\mathbf{v}_k^-\mathbf{v}_j^{+\prime}+(\beta_R-i\beta_I)\,\mathbf{v}_k^+\mathbf{v}_j^{-\prime}\right]+\right.\\ &\left.+\frac{\cos(\omega_k)}{\sin(\omega_k)}\left[\frac{(\beta_R+i\beta_I)}{-2i}\mathbf{v}_k^-\mathbf{v}_j^{+\prime}+\frac{(\beta_R+i\beta_I)}{2i}\mathbf{v}_k^+\mathbf{v}_j^{-\prime}\right]\right\}.\end{aligned} \tag{65}$$

With (65) and (57) and following the lines of the proof of the previous lemmas de result reported in (59) is easily obtained.

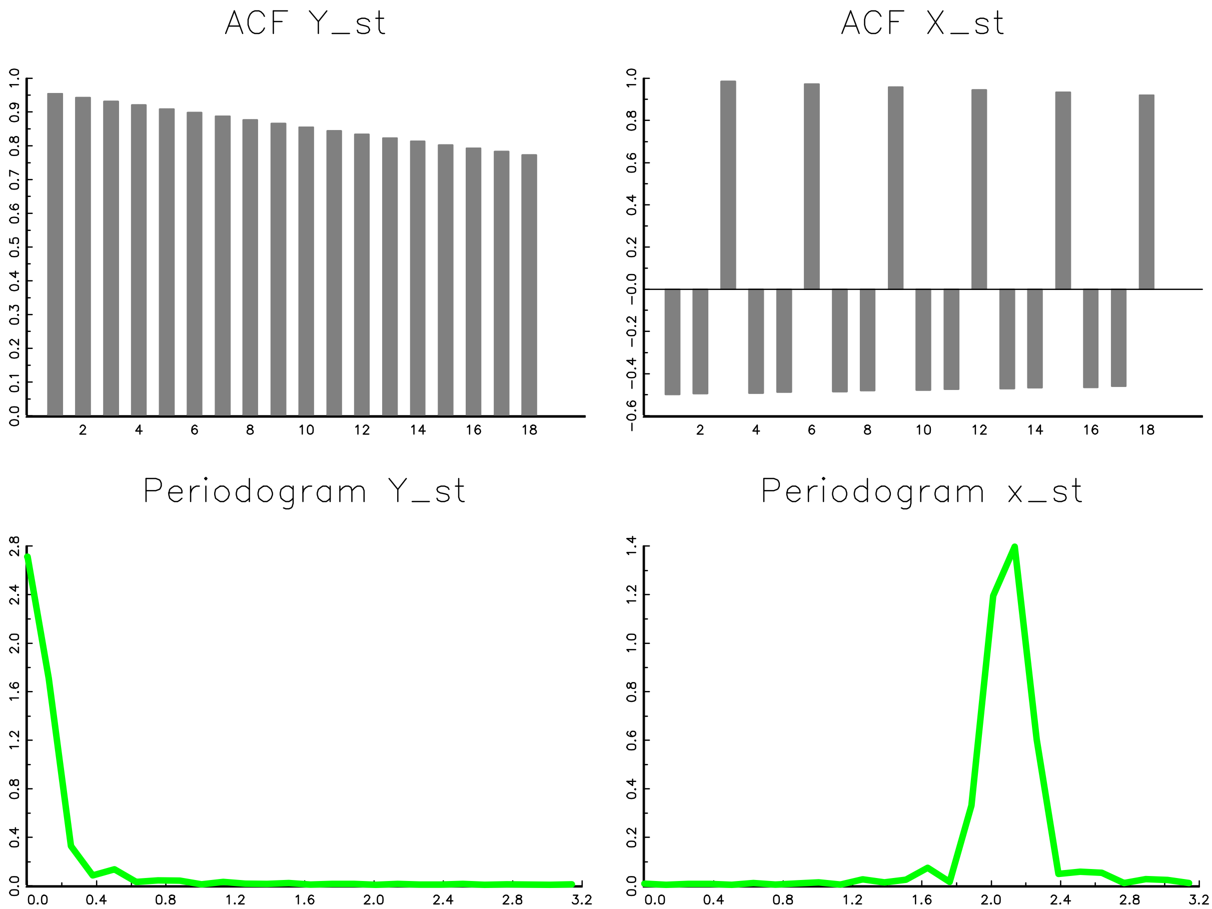
Table 1: Results for case (16) with $\omega_j = \pi/3$.



Johansen Vector of Seasons												
T	$q_0 = 12$	$q_0 = 11$	$q_0 = 10$	$q_0 = 9$	$q_0 = 8$	$q_0 = 7$	$q_0 = 6$	$q_0 = 5$	$q_0 = 4$	$q_0 = 3$	$q_0 = 2$	$q_0 = 1$
200	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0.0592	0.0050

Johansen Demodulated				
	$e^{i\frac{\pi}{3}t}$		$e^{-i\frac{\pi}{3}t}$	
T	$q_0 = 2$	$q_0 = 1$	$q_0 = 2$	$q_0 = 1$
200	0.9942	0.0398	0.9942	0.0398

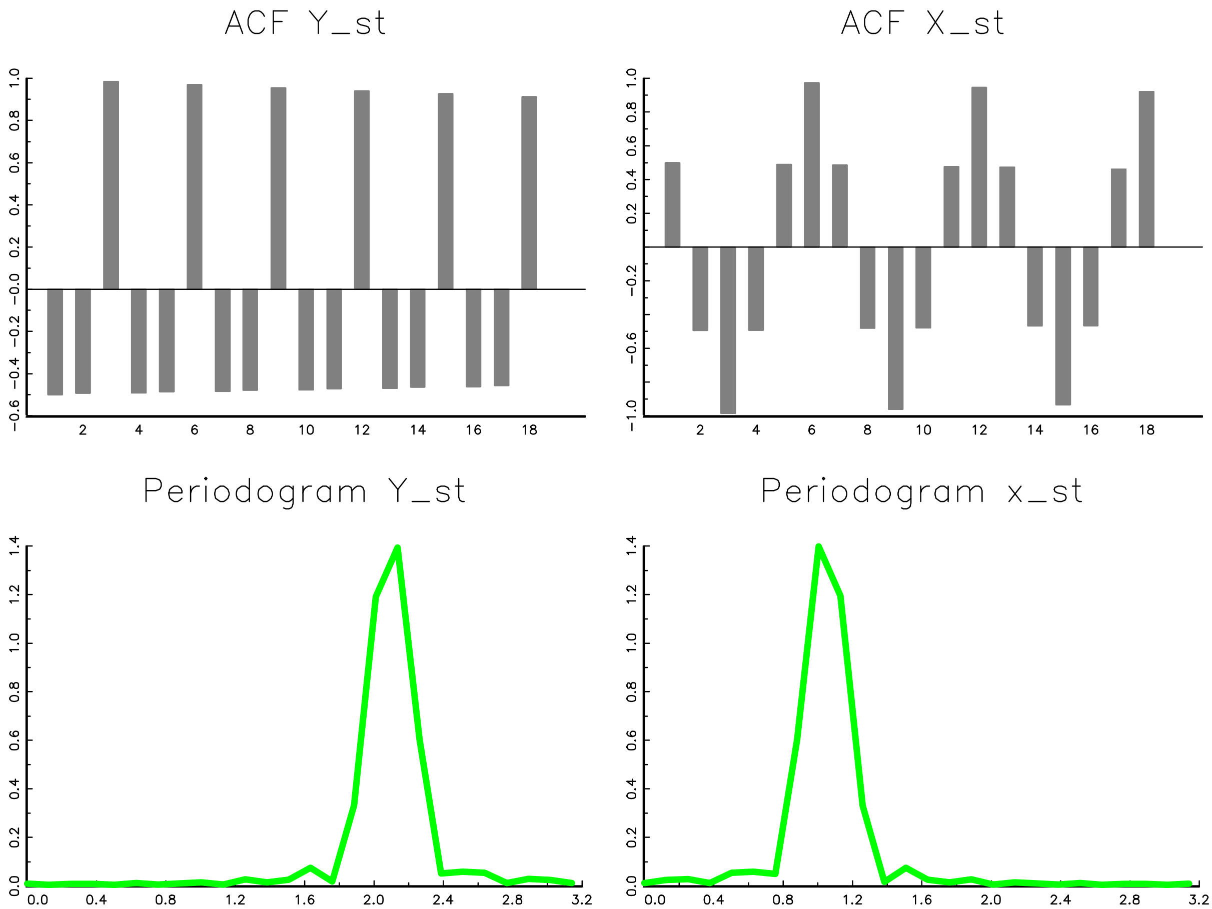
Table 2: Results for case (16) with $\omega_j = 2\pi/3$.



Johansen Vector of Seasons												
T	$q_0 = 12$	$q_0 = 11$	$q_0 = 10$	$q_0 = 9$	$q_0 = 8$	$q_0 = 7$	$q_0 = 6$	$q_0 = 5$	$q_0 = 4$	$q_0 = 3$	$q_0 = 2$	$q_0 = 1$
200	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0586	0.0050
100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9970	0.7448	0.0528	0.0052

Johansen Demodulated				
	$e^{i2\frac{\pi}{3}t}$		$e^{-i2\frac{\pi}{3}t}$	
T	$q_0 = 2$	$q_0 = 1$	$q_0 = 2$	$q_0 = 1$
200	0.9950	0.0462	0.9950	0.0462
100	1.0000	0.0354	1.0000	0.0354

Table 3: Results for case (**??**) with $\omega_k = 2\pi/3$ and $\omega_j = \pi/3$



Johansen Vector of Seasons												
T	$q_0 = 12$	$q_0 = 11$	$q_0 = 10$	$q_0 = 9$	$q_0 = 8$	$q_0 = 7$	$q_0 = 6$	$q_0 = 5$	$q_0 = 4$	$q_0 = 3$	$q_0 = 2$	$q_0 = 1$
200	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0606	0.0028
100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9970	0.7448	0.0528	0.0052

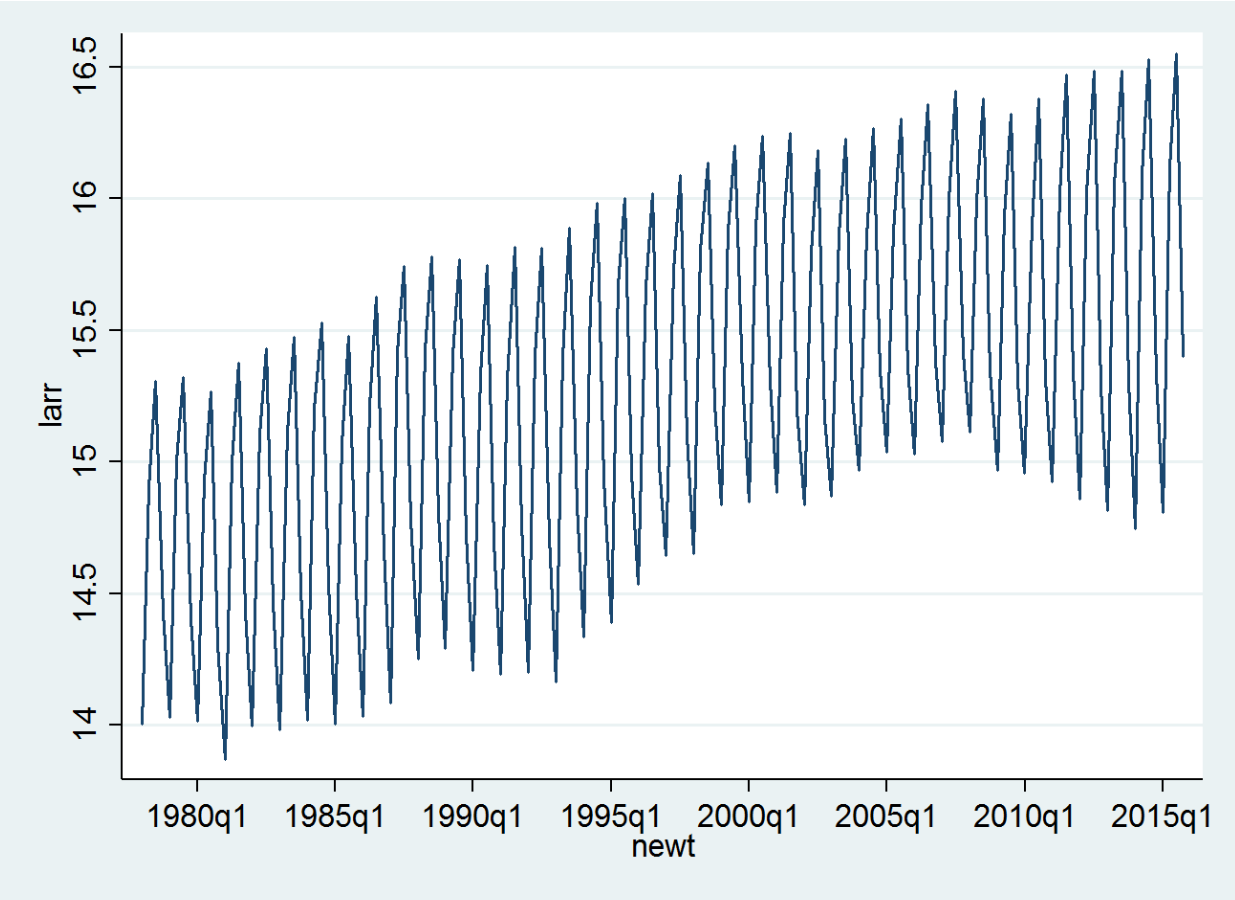
T	Johansen Demodulated			
	$e^{i\frac{\pi}{3}t}$ and $e^{i2\frac{\pi}{3}t}$		$e^{-i\frac{\pi}{3}t}$and $e^{-i2\frac{\pi}{3}t}$	
	$q_0 = 2$	$q_0 = 1$	$q_0 = 2$	$q_0 = 1$
200	1.0000	0.0324	1.0000	0.0324
100	1.0000	0.0354	1.0000	0.0354

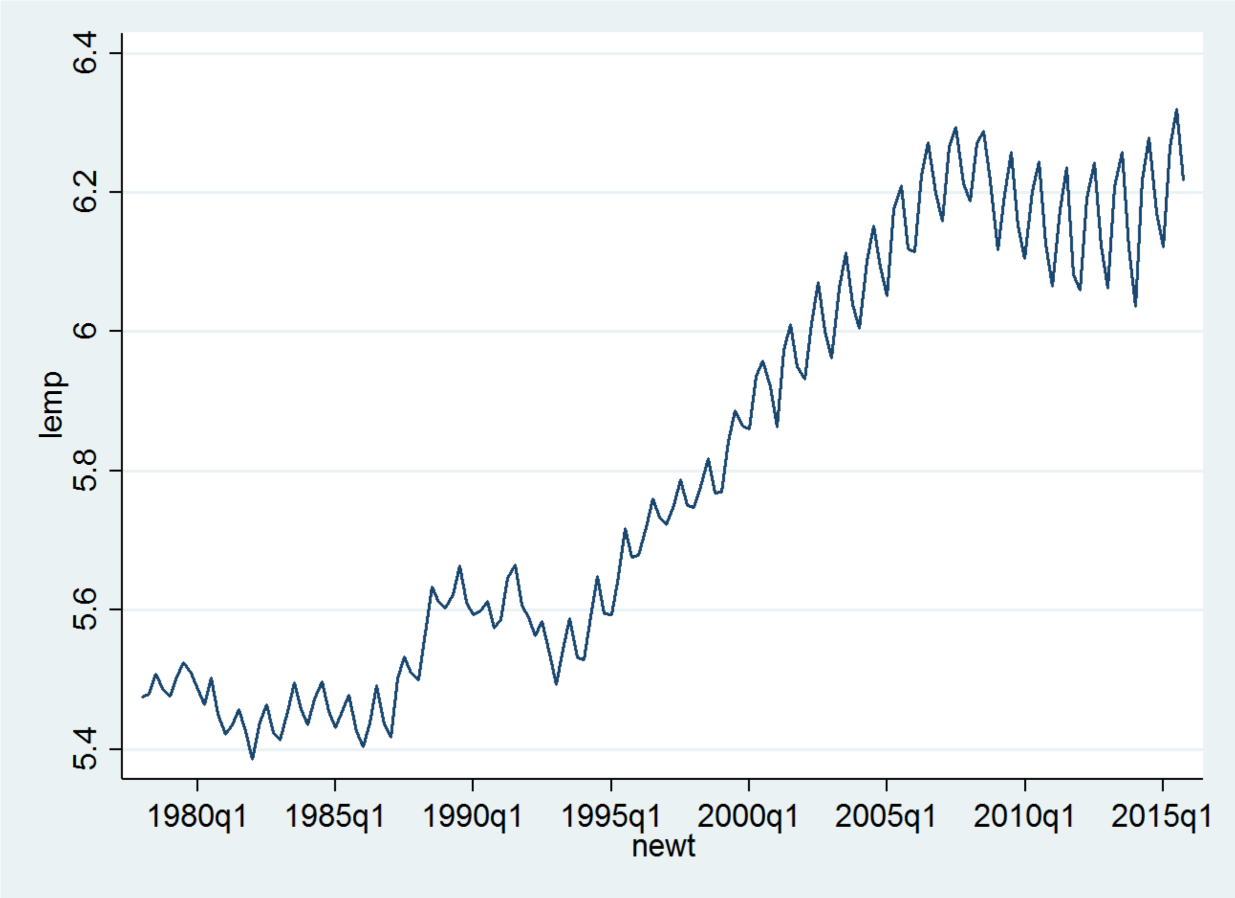
Table 4: Results for cases (25) and (26)

	Johansen Vector of Seasons											
	$q_0 = 12$	$q_0 = 11$	$q_0 = 10$	$q_0 = 9$	$q_0 = 8$	$q_0 = 7$	$q_0 = 6$	$q_0 = 5$	$q_0 = 4$	$q_0 = 3$	$q_0 = 2$	$q_0 = 1$
(25)	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0666	0.0036	0.0000
(26)	1.0000	1.0000	1.0000	1.0000	1.0000	0.9988	0.9478	0.6140	0.0834	0.0078	0.0008	0.0000

	Demodulated $e^{i2\frac{\pi}{3}t}$		Demodulated $e^{-i2\frac{\pi}{3}t}$	
	$q_0 = 2$	$q_0 = 1$	$q_0 = 2$	$q_0 = 1$
(25)	0.0656	0.0048	0.0656	0.0048

	Demod. $e^{i2\frac{\pi}{3}t}$ and $e^{i\frac{\pi}{3}t}$		Demod. $e^{-i2\frac{\pi}{3}t}$ and $e^{-i\frac{\pi}{3}t}$	
	$q_0 = 2$	$q_0 = 1$	$q_0 = 2$	$q_0 = 1$
(26)	0.028	0.0032	0.028	0.0032


16.5
16
15.5
15
14.5
14
larr
1980q1
1985q1
1990q1
1995q1
2000q1
2005q1
2010q1
2015q1
newt


6.4
6.2
6
5.8
5.6
5.4
lemp
1980q1
1985q1
1990q1
1995q1
2000q1
2005q1
2010q1
2015q1
newt