Cointegration and adjustment in the CVAR(∞) representation of some partially observed CVAR(1) models

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Abstract

A multivariate CVAR(1) model for some observed variables and some unobserved variables is analysed using its infinite order CVAR representation of the observations. Cointegration and adjustment coefficients in the infinite order CVAR are found as functions of the parameters in the CVAR(1) model. Conditions for weak exogeneity for the cointegrating vectors in the approximating finite order CVAR are derived. The results are illustrated by a two simple examples of relevance for modelling causal graphs.

Keywords: Adjustment coefficients, cointegrating coefficients, CVAR, causal models.

JEL Classification: C32.

1 Introduction

In a survey paper on long-run causal order, Hoover (2018) applies the CVAR(1) model for the processes $X_t = (x_{1t}, \dots, x_{pt})$ and $T_t = (T_{1t}, \dots, T_{mt})$, to model a causal graph. The model is given by

$$\begin{aligned} \Delta X_{t+1} &= M X_t + C T_t + \varepsilon_{t+1}, \\ \Delta T_{t+1} &= \eta_{t+1}. \end{aligned} \qquad t = 0, \dots, n-1 \tag{1}$$

Here the entry $M_{ij} \neq 0$ means that x_j causes x_i, which is written $x_j \to x_i$, and $C_{ij} \neq 0$ means that $T_j \to x_i$, and it is further assumed that $M_{ii} \neq 0$. Note that the model assumes that there are no causal links from X_t to T_t, so that T_t is strongly exogenous.

A simple example for three variables x_1, x_2, x_3 and a trend T, is the graph

$$T \to x_1 \to x_2 \to x_3,$$

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where the matrices are given by

$$M = \begin{pmatrix} * & * & 0 \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}, C = \begin{pmatrix} * \\ 0 \\ 0 \end{pmatrix}$$

where $*$ indicates a nonzero coefficient.

The paper by Hoover gives a detailed and general discussion of the problems of recovering causal structures from nonstationary observations X_t, or subsets of X_t, when T_t is unobserved, that is, $X_t = (X_{1t}', X_{2t}')'$ where the observations X_{1t} are p_1-dimensional and the unobserved processes X_{2t} and T_t are p_2- and m-dimensional respectively, $p = p_1 + p_2$.

Model (1) is therefore rewritten as

$$\begin{aligned} \Delta X_{1t+1} &= M_{11}X_{1t} + M_{12}X_{2t} + C_1 T_t + \varepsilon_{1t+1}, \\ \Delta X_{2t+1} &= M_{21}X_{1t} + M_{22}X_{2t} + C_2 T_t + \varepsilon_{2t+1}, \\ \Delta T_{t+1} &= \eta_{t+1}. \end{aligned} \tag{2}$$

Note that there is now a causal link from the observed process X_{1t} to the unobserved processes, if $M_{21} \neq 0$.

The process X_{1t} is a linear transformation of $\{X_t, T_t\}$ and therefore allows a CVAR(∞) representation, see for instance Johansen and Juselius (2014),

$$\Delta X_{1t+1} = \alpha\beta' X_{1t} + \sum_{i=1}^{\infty} \Gamma_i \Delta X_{1,t+1-i} + \nu_{t+1}, \tag{3}$$

where $\nu_{t+1} = X_{1t+1} - E_t(X_{1t+1}|X_{1s}, s \leq t)$ is the prediction error for the observation X_{1t+1} given the past, and the coefficients Γ_i decrease exponentially. It follows under the usual $I(1)$ conditions, see Johansen (1996, Theorem 4.2), that X_{1t} has the Granger representation

$$X_{1t} = \beta_\perp(\alpha_\perp'\Gamma\beta_\perp)^{-1}\alpha_\perp' \sum_{i=1}^{t} \nu_i + Y_t + A, \tag{4}$$

where Y_t is a stationary process, $\Gamma = I_{p_1} - \sum_{i=1}^{\infty}\Gamma_i$, and A depends on initial values and $\beta' A = 0$.

A statistical analysis, including estimation of α, β, and Γ, can be conducted for the observations X_{1t} using an approximating finite order CVAR, see Saikkonen (1992) and Saikkonen and Lütkepohl (1996).

Hoover (2018) investigates in particular if weak exogeneity for β in the approximating finite order CVAR, that is, a zero row in α, is a useful tool for finding the causal structure in the graph.

The present note solves the problem of finding expressions for the parameters α and β in the CVAR(∞) model (3) for the observation X_{1t}, as functions of the parameters in model (2), and finds conditions on these for the presence of a zero row in α, and hence weak exogeneity for β in the approximating finite order CVAR.

2 The assumptions and main results

First some definitions and assumptions are given. Then the main results on α and β are presented and proved in Theorems (2) and (1). These results rely on Theorem 3 on the solution of an algebraic Riccati equations which is given and proved in the Appendix.

In the following a $k \times k$ matrix is called stable if the eigenvalues are contained in the open unit disk. If A is a $k_1 \times k_2$ matrix of rank $k \leq \min(k_1, k_2)$, an orthogonal complement, $A_\perp$, is defined as a $k_1 \times (k_1 - k)$ matrix of rank $k_1 - k$ for which $A'_\perp A = 0$. If $k_1 = k$, $A_\perp = 0$. Note that $A_\perp$ is only defined up to multiplication from the right by a $(k_1 - k) \times (k_1 - k)$ matrix of full rank. Throughout $E_t(.)$ and $Var_t(.)$ denote conditional expectation and variance given the information $\{X_{1s}, 0 \leq s \leq t\}$. The norm of the $k \times k$ matrix A is $||A|| = \lambda_{\max}^{1/2}(A'A)$, where $\lambda_{\max}(A'A)$ is the largest eigenvalue of $A'A$.

Assumption 1 *Let $\{X_{1t}, X_{2t}, T_t\}$, $0 = 1, \ldots, n$, be given by (2) with starting values zero: $T_0 = 0$, $X_{10} = 0$, $X_{20} = 0$. It is assumed that*

(i) ε_{1t}, ε_{2t}, and η_t are mutually independent and i.i.d. Gaussian with mean zero and variances Ω_1, Ω_2, and Ω_η, where Ω_1 and Ω_2 are diagonal matrices,

(ii) $I_{p_1} + M_{11}$, $I_{p_2} + M_{22}$ and $I_p + M$ are stable,

(iii) $C_{1.2} = C_1 - M_{12}M_{22}^{-1}C_2$ has full rank m.

Assumption 1(ii) on M_{11}, M_{22} and M is taken from Hoover (2018) to ensure that for instance the process X_t given by the equations $X_t = (I_p + M)X_{t-1} + input$, is stationary if the input is stationary, such that the nonstationarity of X_t in model (2) is created by the trends T_t, and not by the own dynamics of X_t as given by M. It follows from this assumption that M is nonsingular, because $I_p + M$ is stable, and similarly for M_{11} and M_{22}. Moreover $M_{11.2} = M_{11} - M_{12}M_{22}^{-1}M_{21}$ is nonsingular because

$$\det M = \det M_{22} \det M_{11.2} \neq 0.$$

Assumption 1(iii) on the rank of $C_{1.2} = C_1 - M_{12}M_{22}^{-1}C_2$, ensures that $p_1 \geq m$ and that the m unobserved trends can be estimated from observations of the p_1-dimensional process X_{1t}.

The first result on β is a simple consequence of model (2).

Theorem 1 *Assumption 1 implies that the cointegrating rank is $r = p_1 - m$, and the coefficients β and $\beta_\perp$ in the CVAR(∞) representation, (3), of X_{1t} are given for $p_1 > m$ as*

$$\beta_\perp = M_{11.2}^{-1}C_{1.2} \text{ and } \beta = M'_{11.2}(C_{1.2})_\perp. \tag{5}$$

For $p_1 = m$, $\beta_\perp$ has rank p_1, and there is no cointegration: $\alpha = \beta = 0$.

Proof of Theorem of 1. From the model equations (2) it follows, by eliminating X_{2t} from the first two equations, that

$$\Delta X_{1t+1} - M_{12}M_{22}^{-1}\Delta X_{2t+1} = M_{11.2}X_{1t} + C_{1.2}T_t + \varepsilon_{1t+1} - M_{12}M_{22}^{-1}\varepsilon_{2t+1}.$$

Solving for the nonstationary terms gives

$$M_{11.2}X_{1t} + C_{1.2}T_t = \Delta X_{1t+1} - M_{12}M_{22}^{-1}\Delta X_{2t+1} - \varepsilon_{1t+1} + M_{12}M_{22}^{-1}\varepsilon_{2t+1}. \tag{6}$$

Multiplying by $\beta' M_{11.2}^{-1}$, it is seen that for $\beta' X_{1t}$ to be asymptotically stationary, it must hold that $\beta' M_{11.2}^{-1} C_{1.2} = 0$. By Assumption 1(i), $C_{1.2}$ has rank m, so that β has rank $p_1 - m$, which proves (5). ■

The result for α is more involved and is given in Theorem 2. The proof is a further analysis of (6) and involves using the representation X_{1t} as the sum of prediction errors $\nu_t = X_{1t} - E(X_{1t}|X_{1s}, s < t)$, see (4), and a representation of $E_t(T_t)$ as a (weighted) sum of the prediction errors $\nu_{0t} = X_{1t} - E(X_{1t}|X_{1s}, 0 \leq s < t)$. This requires a result from control theory on the solution of an algebraic Riccati equation together with some results based on the Kalman filter for the calculation of $E_t(T_t)$ and $Var_t(T_t)$. These are collected as Theorem 3 in the Appendix.

For the discussion of these results, it is useful to reformulate (2) by defining the unobserved variables and errors

$$T_t^* = \begin{pmatrix} X_{2t} \\ T_t \end{pmatrix}, \; \eta_t^* = \begin{pmatrix} \varepsilon_{2t} \\ \eta_t \end{pmatrix},$$

and the matrices

$$Q^* = \begin{pmatrix} I_{p_2} + M_{22} & C_2 \\ 0 & I_m \end{pmatrix}, \; M_{21}^* = \begin{pmatrix} M_{21} \\ 0 \end{pmatrix}, \; C^* = (M_{12}; C_1). \tag{7}$$

Then (2) becomes

$$\begin{aligned} X_{1t+1} &= (I_{p_1} + M_{11}) X_{1t} + C^* T_t^* + \varepsilon_{1t+1}, \\ T_{t+1}^* &= M_{21}^* X_{1t} + Q^* T_t^* + \eta_{t+1}^*. \end{aligned} \tag{8}$$

One can then show, see Theorem 3, that based on properties of the Gaussian distribution, a recursion can be found for the calculation of $Var_t(T_t^*)$ and $E_t(T_t^*)$

$$V_{t+1} = Q^* V_t Q^{*\prime} + \Omega^* - Q^* V_t C^{*\prime} (C^* V_t C^{*\prime} + \Omega_1)^{-1} C^* V_t Q^{*\prime}, \tag{9}$$

$$E_{t+1} = M_{21}^* X_{1t} + Q^* E_t + Q^* V_t C^{*\prime} (C^* V_t C^{*\prime} + \Omega_1)^{-1} \nu_{0t+1}. \tag{10}$$

It then follows from results from control theory, that $V = \lim_{t\to\infty} Var_t(T_t^*)$ exists and satisfies the algebraic Riccati equation

$$V = Q^* V Q^{*\prime} + \Omega^* - Q^* V C^{*\prime} (C^* V C^{*\prime} + \Omega_1)^{-1} C^* V Q^{*\prime}. \tag{11}$$

Moreover, the prediction errors $\nu_{0t} = X_{1t} - E_{t-1}(X_{1t})$ are independent $N_{p_1}(0, \Sigma_t)$ for $\Sigma_t = C^* V_t C^{*\prime} + \Omega_1$, and the prediction errors $\nu_t = X_{1t} - E(X_{1t}|X_{1s}, s \leq t-1)$ are independent $N_{p_1}(0, \Sigma)$ for $\Sigma = C^* V C^{*\prime} + \Omega_1$. Finally $E_t(T_t)$ has the representation in the prediction errors, ν_{0t},

$$E_t(T_t) = (0; I_m) \sum_{i=1}^{t} V_i C^{*\prime} \Sigma_i^{-1} \nu_{0i}. \tag{12}$$

Comparing the representation (4) for X_t and (12) for $E_t(T_t)$ gives a more precise relation between the coefficients of the nonstationary terms in (6). The main result of the paper is to show how this result leads to expressions for the coefficients α and $\alpha_\perp$ as functions of the parameters in model (2).

Theorem 2 *Assumption 1 implies that the coefficients* α *and* $\alpha_\perp$ *in the CVAR(*∞*) representation of* X_{1t} *are given for* $p_1 > m$ *as*

$$\alpha_\perp = \Sigma^{-1}(M_{12}V_{2T} + C_1 V_{TT}),\ \ \alpha = \Sigma(M_{12}V_{2T} + C_1 V_{TT})_\perp, \tag{13}$$

where

$$\Sigma = Var(\nu_t) = C^* V C^{*\prime} + \Omega_1 = (M_{12}; C_1)\begin{pmatrix} V_{22} & V_{2T} \\ V_{T2} & V_{TT} \end{pmatrix}(M_{12}; C_1)' + \Omega_1. \tag{14}$$

Proof of Theorem 2. The left hand side of (6) has two nonstationary terms. The observations X_{1t} is represented in (4) as a random walk in the prediction errors ν_i, and T_t is a random walk in η_i. Taking conditional expectation given $X_{1s}, 0 \leq s \leq t$, T_t is replaced by $E_t(T_t)$, which in (12) is represented as a weighted sum of ν_{0i}. Thus taking conditional expectation in (6) gives

$$M_{11.2}X_{1t} + C_{1.2}E_t(T_t) = E_t(\Delta X_{1t+1} - M_{12}M_{22}^{-1}\Delta X_{2t+1}), \tag{15}$$

where the right hand side is bounded in the mean:

$$E|E_t(\Delta X_{1t+1} - M_{12}M_{22}^{-1}\Delta X_{2t+1})| \leq c\{E|\Delta X_{1t+1}| + |\Delta X_{2t+1}|\} \leq c.$$

Setting $t = [nu]$ and dividing by $n^{1/2}$, it follows from (4) that

$$n^{-1/2}X_{1[nu]} \overset{\mathrm{D}}{\to} \beta_\perp(\alpha_\perp'\Gamma\beta_\perp)^{-1}\alpha_\perp' W_\nu(u), \tag{16}$$

where $W_\nu(u)$ the Brownian motion generated by the prediction errors ν_t.

From (12) it follows similarly that

$$n^{-1/2}E_{[nu]}(T_{[nu]}) = (0; I_m)n^{-1/2}\sum_{t=1}^{[nu]} V_t C^{*\prime}\Sigma_t^{-1}\nu_{0t} \overset{\mathrm{D}}{\to} (0; I_m)VC^{*\prime}\Sigma^{-1}W_\nu(u). \tag{17}$$

To see this, replace (V_t, Σ_t) by (V, Σ) and ν_{0t} by ν_t.

The first replacement is valid because for $\delta_t' = V_t C^{*\prime}\Sigma_t^{-1} - VC^{*\prime}\Sigma^{-1} \to 0$, it holds that

$$Var(n^{-1/2}\sum_{t=1}^{[nu]}\delta_t'\nu_{0t}) = T^{-1}\sum_{t=1}^{[nu]}\delta_t'\Sigma_t\delta_t \to 0,\ T \to \infty.$$

The second replacement is valid, using $||\Gamma_i|| \leq c\rho^i$ for some $\rho < 1$, because from (3) it is seen that

$$E(\Delta X_{t+1}|X_{1s}, s \leq t) = \alpha\beta' X_t + \sum_{i=1}^{t}\Gamma_i\Delta X_{t+1-i} + \sum_{i=t+1}^{\infty}\Gamma_i\Delta X_{t+1-i},$$

$$E(\Delta X_{t+1}|X_{1s}, 0 \leq s \leq t) = E_t(\Delta X_{t+1}) = \alpha\beta' X_t + \sum_{i=1}^{t}\Gamma_i\Delta X_{t+1-i} + \sum_{i=t+1}^{\infty}\Gamma_i E_t(\Delta X_{t+1-i}),$$

and hence

$$\nu_{0t+1} - \nu_{t+1} = \sum_{i=t+1}^{\infty} \Gamma_i \{\Delta X_{t+1-i} - E_t(\Delta X_{t+1-i})\}.$$

Then

$$E||n^{-1/2} \sum_{t=1}^{[nu]} VC^{*\prime}\Sigma^{-1}(\nu_{0t} - \nu_t))|| \leq cn^{-1/2} \sum_{t=1}^{[nu]} E||\nu_{0t} - \nu_t|| \leq cn^{-1/2} \sum_{t=1}^{[nu]} \sum_{i=t+1}^{\infty} \rho^i \to 0.$$

Finally, setting $t = [nu]$ and normalizing (15) by $n^{-1/2}$, it follows that

$$M_{11.2}\beta_\perp(\alpha_\perp'\Gamma\beta_\perp)^{-1}\alpha_\perp' W_\nu(u) + C_{1.2}(0; I_m)VC^{*\prime}\Sigma^{-1}W_\nu(u) = 0 \text{ for } u \in [0,1].$$

This relation shows that the coefficient to $W_\nu(u)$ is zero, so that $\alpha_\perp$ can be chosen as

$$\alpha_\perp = \Sigma^{-1}C^*V(0; I_m)' = \Sigma^{-1}(M_{12}V_{2T} + C_1 V_{TT})$$

and therefore $\alpha = \Sigma(M_{12}V_{2T} + C_1V_{TT})_\perp$ which proves (13). ■

Thus, in order to investigate a zero row in α, the matrix V is needed. This is easy to calculate from the recursion of V_t for given value of the parameters, but the properties of V are more difficult to evaluate. In general α does not contain a zero row, but if $M_{12}V_{2T} = 0$, the expressions for α and $\alpha_\perp$ simplify, so that simple conditions on M_{12} and C_1 imply a zero row in α and hence gives weak exogeneity in the statistical analysis of the approximating finite order CVAR. This extra condition, $M_{12}V_{2T} = 0$, implies that

$$\Sigma = (M_{12}; C_1)V(M_{12}; C_1)' + \Omega_1 = M_{12}V_{22}M_{12}' + C_1V_{TT}C_1' + \Omega_1,$$

and

$$(M_{12}V_{2T} + C_1V_{TT})_\perp = (C_1V_{TT})_\perp = C_{1\perp},$$

such that α simplifies to

$$\alpha = (M_{12}V_{22}M_{12}' + C_1V_{TT}C_1' + \Omega_1)C_{1\perp} = (M_{12}V_{22}M_{12}' + \Omega_1)C_{1\perp}.$$

Thus, a condition for a zero row in α is

$$e_i'\alpha = e_i'M_{12}V_{22}M_{12}'C_{1\perp} + \omega_i e_i'C_{1\perp} = 0 \tag{18}$$

because $\Omega_1 = \text{diag}(\omega_1, \ldots, \omega_{p_1})$. This is simple to check by inspecting the matrices M_{12} and $C_{1\perp}$ in model (2). In the next section two cases are given, where such a simple solution is available.

3 Two examples of simplifying assumptions

Case 1 ($M_{12} = 0$)
If the unobserved process X_{2t} does not cause the observation X_{1t}, then $M_{12} = 0$. Therefore $M_{12}V_{2T} = 0$ and from (18) it follows that

$$e_i'\alpha = \omega_i e_i'C_{1\perp} = 0.$$

Thus, α has a zero row if $C_{1\perp}$ has a zero row.

An example of $M_{12} = 0$ is the chain $T \to x_1 \to x_2 \to x_3$, where $X_1 = \{x_1, x_2, x_3\}$ is observed and $X_2 = 0$, and hence $M_{12} = 0$ and $C_2 = 0$. Then, because $T \to x_1$

$$C_1 = \begin{pmatrix} * \\ 0 \\ 0 \end{pmatrix}, \; C_{1\perp} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, the first row of $C_{1\perp}$ is a zero row, such that x_1 is weakly exogenous.

To formulate the next case, a definition of strong orthogonality of two matrices is introduced.

Definition 1 *Let A be a $k \times k_1$ matrix and B a $k \times k_2$ matrix. Then A and B are called strongly orthogonal if $A'DB = 0$ for all diagonal matrices D, or equivalently if $A_{ji}B_{j\ell} = 0$ for all i, j, ℓ.*

Thus, in particular if M_{12} and C_1 are strongly orthogonal, and if T causes a variable in X_1, then X_2 does not cause that variable. The expression for V simplifies in the following case.

Lemma 1 *If $C_2 = 0$, and $M_{12}'\Omega_1^{-1}C_1 = 0$, then $Q^* = \mathrm{blockdiag}(I_{p_2} + M_{22};\, I_m)$, and $V_{2T} = 0$ such that $V = \mathrm{blockdiag}(V_{22};\ V_{TT})$.*

Proof of Lemma 1. The proof is by induction and the result holds for $t = 0$, where $V_0 = 0$. Assume therefore that $V_t =$blockdiag$(V_{t22};\, V_{tTT})$ and consider the expression for V_{t+1}, see (9). In this expression Q^* is block diagonal (because $C_2 = 0$) and $Q^*V_tQ^{*\prime}$ and Ω^* are block diagonal, and the same holds for $Q^*V_t^{1/2}$. Thus, it is enough to show that

$$V_t^{1/2}C^{*\prime}\{C^*V_tC^{*\prime} + \Omega_1\}^{-1}C^*V_t^{1/2},$$

is block diagonal. To simplify the notation define the normalized matrices

$$\check{M} = \Omega_1^{-1/2}M_{12}V_{t22}^{1/2} \text{ and } \check{C} = \Omega_1^{-1/2}C_1V_{tTT}^{1/2}.$$

Then, by assumption,

$$\check{M}'\check{C} = V_{t22}^{1/2}M_{12}'\Omega_1^{-1}C_1V_{tTT}^{1/2} = 0,$$

so that, using $V_{t2T} = 0$,

$$V_t^{1/2}C^{*\prime}(C^*V_tC^{*\prime} + \Omega_1)^{-1}C^*V_t^{1/2} = (\check{M}, \check{C})'(\check{M}\check{M}' + \check{C}\check{C}' + I_{p_1})^{-1}(\check{M}, \check{C}).$$

A direct calculation shows that

$$(\check{M}\check{M}' + \check{C}\check{C}' + I_{p_1})^{-1} = I_{p_1} - \check{M}(I_{p_2} + \check{M}'\check{M})^{-1}\check{M}' - \check{C}(I_{p_2} + \check{C}'\check{C})^{-1}\check{C}',$$

and that

$$\check{M}'\{I_{p_1} - \check{M}(I_{p_2} + \check{M}'\check{M})^{-1}\check{M}' - \check{C}(I_{p_2} + \check{C}'\check{C})^{-1}\check{C}'\}\check{C} = 0$$

such that $(\check{M}, \check{C})'(\check{M}\check{M}' + \check{C}\check{C}' + I_{p_1})^{-1}(\check{M}, \check{C})$ is block diagonal.

Then, $V_t^{1/2}C^{*\prime}\{C^*V_tC^{*\prime} + \Omega_1\}^{-1}C^*V_t^{1/2}$ and hence V_{t+1} and V are block diagonal. ∎

Case 2 ($C_2 = 0$, and M_{12} and C_1 are strongly orthogonal)
Because $C_2 = 0$ and $M'_{21}\Omega_1^{-1}C_1 = 0$, Lemma 1 shows that $V_{2T} = 0$, so that the condition $M_{12}V_{2T} = 0$ and (18) hold. Moreover, strong orthogonality also implies that $M'_{12}C_1 = 0$ such that $M_{12} = C_{1\perp}\xi$ for some ξ. Hence

$$e'_i\alpha = e'_iM_{12}V_{22}M'_{12}C_{1\perp} + \omega_ie'_iC_{1\perp} = e'_iC_{1\perp}(\xi V_{22}M'_{12}C_{1\perp} + \omega_iI_{p_1-m}), \tag{19}$$

and therefore, a zero row in $C_{1\perp}$ gives a zero row in α.

Consider again the chain $T \to x_1 \to x_2 \to x_3$, but assume now that x_2 is not observed. Thus, $X_1 = \{x_1, x_3\}$ and $X_2 = \{x_2\}$. Here T causes x_1, and x_2 causes x_3, so that

$$M_{12} = \begin{pmatrix} 0 \\ * \end{pmatrix}, \; C_1 = \begin{pmatrix} * \\ 0 \end{pmatrix}, \; C_2 = 0.$$

Note that $M'_{12}DC_1 = 0$ for all diagonal D because T and X_2 cause disjoint subsets of X_1. This together with $C_2 = 0$ implies that V is block diagonal and that (19) holds such that $e'_i\alpha = 0$ if

$$e'_iC_{1\perp} = e'_i\begin{pmatrix} 0 \\ * \end{pmatrix} = 0.$$

Thus x_1 is weakly exogenous.

4 Conclusion

This paper investigates the problem of finding adjustment and cointegrating coefficients for the infinite order CVAR representation of a partially observed simple CVAR(1) model. The main tools are some classical results for the solution of the algebraic Riccati equation, and the results are exemplified by an analysis of CVAR(1) models for causal graphs in two cases where simple conditions for weak exogeneity are derived in terms of the parameters of the CVAR(1) model.

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6 Appendix

The next Theorem shows how the Kalman filter can be used to calculate $Var_t(T_t^*)$ and $E_t(T_t^*)$ using the same technique as for the common trends model and proves the existence of the limit of V_t. The last result follows from the theory of the algebraic Riccati equation, see Lancaster and Rodman (1995), in the following LR(1995).

Theorem 3 *Let X_{1t} and T_t^* be given by model (8) and let Assumption 1 be satisfied. Then $V_t = Var_t(T_t^*)$ and $E_t = E_t(T_t^*)$ are given recursively, using the starting values $E_0 = 0$, $V_0 = 0$, by*

$$V_{t+1} = Q^*V_tQ^{*\prime} + \Omega^* - Q^*V_tC^{*\prime}\Sigma_t^{-1}C^*V_tQ^{*\prime}, \tag{20}$$

$$E_{t+1} = M_{21}^*X_{1t} + Q^*E_t + Q^*V_tC^{*\prime}\Sigma_t^{-1}\nu_{0t+1}, \tag{21}$$

where
$$\Sigma_t = C^* V_t C^{*\prime} + \Omega_1, \tag{22}$$
and the prediction errors
$$\nu_{0t+1} = X_{1t+1} - E_t(X_{1t+1}) \tag{23}$$
are independent $N_{p_1}(0, \Sigma_t)$.

The sequence V_t *is nondecreasing and converges to a finite positive limit* V*, which satisfies the algebraic Riccati equation,*

$$V = Q^* V Q^{*\prime} + \Omega^* - Q^* V C^{*\prime} \Sigma^{-1} C^* V Q^{*\prime}, \quad \Sigma = C^* V C^{*\prime} + \Omega_1. \tag{24}$$

Furthermore
$$Q^* - Q^* V C^{*\prime} \Sigma^{-1} C^* \tag{25}$$
is stable, and $E_t(T_t)$ *satisfies the equation*

$$E_{t+1}(T_{t+1}) = E_t(T_t) + (0; I_m) V_t C^{*\prime} \Sigma_t^{-1} \nu_{0t+1}. \tag{26}$$

Proof of Theorem 3. The variance $V_t = Var_t(T_t^*)$ can be calculated recursively, using the properties of the Gaussian distribution, as

$$\begin{aligned} Var_{t+1}(T_{t+1}^*) &= Var_t(T_{t+1}^* | X_{1t+1}) \\ &= Var_t(T_{t+1}^*) - Cov_t(T_{t+1}^*; X_{1t+1}) Var_t(X_{1t+1})^{-1} Cov_t(X_{1t+1}; T_{t+1}^*). \end{aligned} \tag{27}$$

From the model equations (8), it follows that

$$Var_t(T_{t+1}^*) = Var_t\{M_{21}^* X_{1t} + Q^* T_t^* + \eta_{t+1}^*\} = Q^* Var_t(T_t^*) Q^{*\prime} + \Omega^*, \tag{28}$$
$$Cov_t(T_{t+1}^*; X_{1t+1}) = Cov_t\{T_{t+1}^*; (I_{p_1} + M_{11}) X_{1t} + C^* T_t^* + \varepsilon_{1t+1}\} = Q^* Var_t(T_t^*) C^{*\prime}, \tag{29}$$
$$Var_t(X_{1t+1}) = Var_t\{(I_{p_1} + M_{11}) X_{1t} + C^* T_t^* + \varepsilon_{1t+1}\} = C^* Var_t(T_t^*) C^{*\prime} + \Omega_1. \tag{30}$$

Then (27)-(30) give the recursion for $V_t = Var_t(T_t^*)$ in (20). Similarly for the conditional mean it is seen that

$$\begin{aligned} E_{t+1}(T_{t+1}^*) &= E_t(T_{t+1}^* | X_{1t+1}) = E_t(T_{t+1}^*) + Cov_t(T_{t+1}^*; X_{1t+1}) Var_t(X_{1t+1})^{-1} \nu_{0t+1}, \\ E_t(T_{t+1}^*) &= M_{21}^* X_{1t} + Q^* E_t(T_t^*), \end{aligned}$$

implies (21) with prediction error $\nu_{0t+1} = X_{1t+1} - E_t(X_{1t+1})$.

Note that (20) is the usual recursion from the Kalman filter equations for the state space model obtained from (8) for $M_{21}^* = 0$, see Durbin and Koopman (2012). Note also, however, that (21) is not the usual recursion from the common trends model, because of the first term containing M_{21}^*. It is seen from (20) that if V_t converges to V, then V has to satisfy the algebraic Riccati equation (24) and Σ given as indicated.

The result that V_t converges to a finite positive limit follows from LR (1995, Lemma 17.5.4), where the assumptions, in the present notation, are

$a.1$ $(Q^*; I_{p_2+m})$ is controllable,
$a.2$ $(Q^*; I_{p_2+m})$ is stabilizable,
$a.3$ $(C^*; Q^*)$ is detectable.

Before giving the proof, some definitions from control theory are given, which are needed for checking the conditions of the results in LR(1995).

Let A be a $k \times k$ matrix and B be a $k \times k_1$ matrix.

$d.1$ The pair $\{A, B\}$ is called *controllable* if

$$rank(B; AB; \ldots; A^{k-1}B) = k,$$

LR(1995, (4.1.3)).

$d.2$ The pair $\{A; B\}$ is *stabilizable* if there is a $k_1 \times k$ matrix K, such that $A + BK$ is stable LR(1995, p. 90, line 5-).

$d.3$ Finally $\{B; A\}$ is *detectable* means that $\{A'; B'\}$ is stabilizable, LR(1995, page 91 line 6-).

The first assumption, $a.1$, is easy to check: The pair $(Q^*; I_{p_2+m})$ is controllable, see $d.1$, means that

$$\mathrm{rank}(I_{p_2+m}; Q^* I_{p_2+m}; \ldots; Q^{*p_2+m-1} I_{p_2+m}) = p_2 + m.$$

The second assumption, $a.2$, follows because controllability implies stabilizability, see LR (1995, Theorem 4.4.2).

Finally $d.3$ shows that $(C^*; Q^*)$ detectable means $(Q^{*\prime}; C^{*\prime})$ stabilizable, and LR(1995, Theorem 4.5.6 (b)), see also Hautus (1969), shows that $(Q^{*\prime}; C^{*\prime})$ is stabilizable, if and only if

$$\mathrm{rank}(Q^{*\prime} - \lambda I_{p_2+m}; C^{*\prime}) = \mathrm{rank}\begin{pmatrix} M_{12} & C_1 \\ I_{p_2} + M_{22} - \lambda I_{p_2} & C_2 \\ 0 & I_m - \lambda I_m \end{pmatrix} = p_2 + m \text{ for all } |\lambda| \geq 1.$$

For $\lambda = 1$, using $C_{1.2} = C_1 - M_{12} M_{22}^{-1} C_2$ and Assumption 1, it follows that

$$\begin{aligned}\mathrm{rank}(M(1)) = \mathrm{rank}\begin{pmatrix} M_{12} & C_1 \\ M_{22} & C_2 \end{pmatrix} &= \mathrm{rank}\begin{pmatrix} 0 & C_{1.2} \\ M_{22} & C_2 \end{pmatrix} \\ &= \mathrm{rank}(C_{1.2}) + \mathrm{rank}(M_{22}) = m + p_2.\end{aligned}$$

For $|\lambda| > 1$, using Assumption 1(ii), it is seen that

$$\mathrm{rank}(M(\lambda)) = \mathrm{rank}(I_{p_2} + M_{22} - \lambda I_{p_2}) + \mathrm{rank}(I_m - \lambda I_m) = p_2 + m,$$

because λ is not an eigenvalue of the stable matrix $I_{p_2} + M_{22}$, when $|\lambda| > 1$.

Thus $(Q^{*\prime}; C^{*\prime})$ is stabilizable, and assumptions $a.1$, $a.2$, $a.3$ hold and LR (1995, Lemma 17.5.4) applies. This proves that limit $V = \lim_{t \to \infty} V_t$ exists and (25) holds.

Multiplying (21) by $(0; I_m)$ it is seen, using $(0; I_m)Q^* = (0; I_m)$, and $(0; I_m)M_{21}^* = 0$, that a recursion for $E_t(T_t)$ is given by (26). ■

7 References

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