1^{st} Partial Exam

[2h]

1. Let $(X_n)_{n\geq 0}$ be Markov(λ, $\mathbf{P}$) with $\mathbf{P}$ irreducible and aperiodic and with invariant distribution π. Show that, for any λ :

   (a) $\mathbb{P}(X_n = j) \to \pi_j \quad$ as $\quad n \to \infty \quad$ for all j ;

   (b) $p_{ij}^{(n)} \to \pi_j \quad$ as $\quad n \to \infty \quad$ for all i, j .

   Then, assuming that

   $$\mathbf{P} = \begin{pmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{pmatrix}$$

   and $\lambda = \begin{pmatrix} 1 & 0 \end{pmatrix}$:

   (c) define explicitly the two dimensional Markov chain that you introduced in the previous proof;

   (d) compute the mean and the tail distribution of the stopping time you used in the previous proof;

   (e) explicitly show that the estimation you used to prove the convergence actually holds.

   (6 Points)

2. A particle moves on the six vertices of a hexagon in the following way: at each step the particle is equally likely to move and stay, then in case of moving it is equally likely to move to each of its two adjacent vertices.

   Let i be the initial vertex occupied by the particle and o the vertex opposite to i .

   (a) Describe the Markov chain model, classify its states and their periods

   and calculate each of the following quantities:

   (b) the expected number of steps until the particle returns to i ;

   (c) the expected number of visits to o until the first return to i ;

   (d) the expected number of steps until the first visit to o .

   (4 Points)