Notes

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Definition of period and cyclic classes

Theorem 1 ([1] 1.8.4). *Let* $\mathbf{P}$ *be irreducible. There is an integer* $d \geq 1$*, called the* period of $\mathbf{P}$*, and a partition*

$$\mathbb{S} = C_0 \cup C_1 \cup \ldots \cup C_{d-1}$$

such that (setting $C_{nd+r} = C_r$*)*

a) $p_{ij}^{(n)} > 0$ *only if* $i \in C_r$ *and* $j \in C_{r+n}$ *for some* r*;*

b) $p_{ij}^{(nd)} > 0$ *for all sufficiently large* n*, for all* $i, j \in C_r$ *for all* r*.*

The sets $(C_r)_{r\in\{0,1,\ldots,d-1\}}$ *are called* cyclic classes of $\mathbf{P}$*.*

Proof. If $\mathbb{S} = \{k\}$, the theorem is trivial with $d = 1$. So we assume there are at least two states in $\mathbb{S}$.

- Fix a $k \in \mathbb{S}$ and let $T_k = \{n : p_{kk}^{(n)} > 0\}$. Since the chain is irreducible, there are $m_1 > 0$ and $m_2 > 0$, such that $p_{ki}^{(m_1)} > 0$ and $p_{ik}^{(m_2)} > 0$, for $i \in \mathbb{S}$, $i \neq k$. Therefore $m_1 + m_2 \in T_k$. In addition T_k is closed under addition, that is, if $n_1, n_2 \in T_k \implies n_1 + n_2 \in T_k$, therefore T_k contains at least all integer multiples of $(m_1 + m_2)$ and it follows that it has infinite elements.
- Let $d = \min\{n_2 - n_1 : n_1, n_2 \in T_k,\, n_2 > n_1\}$. The minimum is attained since $d \in \mathbb{N}$ and $d \geq 1$, so we define $d = n_2^* - n_1^*$ with $n_1^*, n_2^* \in T_k$.
- There exists an N such that for all $n > N$, $n\,d \in T_k$:
  Let $N = n_1^*(n_1^* - 1)$ and take $n > N$. Write $n = qn_1^* + r$ with $r \in \{0, 1, \ldots, n_1^* - 1\}$ and $q \geq n_1^* - 1$. It follows that $nd = qdn_1^* + rd = qdn_1^* + r(n_2^* - n_1^*) = (qd - r)n_1^* + rn_2^*$. Since $qd - r \geq q - r \geq 0$ it follows that $nd \in T_k$ by the closure of T_k under addition and the fact that $n_1^*, n_2^* \in T_k$.
- Let $C_r = \{i \in \mathbb{S} : p_{ki}^{(nd+r)} > 0 \text{ for some } n \geq 0\}$. By irreducibility $\mathbb{S} \subset \bigcup_{r\in\{0,1,\ldots,d-1\}} C_r$. We define $C_{nd+r} = C_r$.
- Since $n\,d \in T_k$ for $n > N$, an alternative and equivalent definition for the set C_r would be

  $$C_r = \{i \in \mathbb{S} : \exists N_{ir} \geq 0 \mid p_{ki}^{(nd+r)} > 0 \text{ for all } n \geq N_{ir}\}$$

- The collection $(C_r)_{r\in\{0,1,\ldots,d-1\}}$ forms a partition of $\mathbb{S}$:
  We have to show that for $r, s \in \{0, 1, \ldots, d-1\}$ and $s \neq r$, $C_r \cap C_s = \emptyset$.
  Assume that there exists an $n \geq \max\{N_{ir}, N_{is}\}$ such that $p_{ki}^{(nd+r)} > 0$ and $p_{ki}^{(nd+s)} > 0$ with $r > s$, and let $m > 0$ be an integer such that $p_{ik}^{(m)} > 0$. We have $p_{kk}^{(nd+r+m)} > 0$ and $p_{kk}^{(nd+s+m)} > 0$ implying that $nd+r+m, nd+s+m \in T_k$, with $(nd+r+m)-(nd+s+m) = r - s < d$, that is absurd by the minimality definition of d.
- Proof of a):
  Suppose $p_{ij}^{(n)} > 0$ and $i \in C_r$ and let $m > 0$ be an integer such that $p_{ki}^{(md+r)} > 0$. Therefore $p_{kj}^{(n+md+r)} > 0$, implying that $j \in C_{n+r}$.
- The period d is the greatest common divisor of the set T_k, i.e. $d = G.C.D.(T_k)$:
  Since $k \in C_0$, by a), the only possible m such that $p_{kk}^{(m)} > 0$ are multiple of the period d, that is $m = nd$. In particular, n_1^*, n_2^* are multiple of d and $T_k \subset \{nd : n \in \mathbb{N}\}$.
- Proof of b):
  Choose N such that for all $h > N$, $h\,d \in T_k$ and let m_1, m_2 be such that $p_{ik}^{(m_1)} > 0$ and $p_{kj}^{(m_2)} > 0$, by irreducibility. Then $p_{ij}^{(m_1+hd+m_2)} \geq p_{ik}^{(m_1)} p_{kk}^{(hd)} p_{kj}^{(m_2)} > 0$ for all $h > N$. Since $i, j \in C_r$, $m_1 + m_2 = Md$, with $M > 0$, that is, the sum is multiple of the period. The result follows for any $n > M + N$.

□

References

[1] Norris, J.R., *Markov Chains.*, Cambridge University Press, 1997.