Notes

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The homogeneous Poisson process

In this section we give a precise definition of what we mean by a homogeneous Poisson process. To do this we first give the definition of a counting process.

Definition 1. *A counting process, $N(t)$, is any integer valued process with the following properties:*

i $N(0) = 0$.

ii $N(t+s) \geq N(t), \quad \forall s \geq 0$.

We will use often the notation $N(t, t+s] \triangleq N(t+s) - N(t)$ to denote the number of point counted in the interval $(t, t+s]$ and in general $N(A)$ to denote the number of points counted in a general set $A \in \mathcal{B}(\mathbb{R}^+)$.

Definition 2. *A* ***homogeneous Poisson process****, $N(t)$, with rate λ is defined as a counting process with independent and stationary increments with the property that the number of points counted in an interval $(t, t+s]$ is given by a Poisson distribution with parameter $\lambda\, s$, i.e.*

$$N(t, t+s] \sim Po(\lambda\, s) \; .$$

The property of independence and stationarity of the increments implies that the number of points counted in any two disjoint intervals is given by two independent Poisson random variable whose parameters are proportional to the size of the corresponding intervals by the proportional constant λ.

Let S_n be the location of the n-th point when $n \geq 1$ and assume that $S_0 \triangleq 0$, we define the n-th inter-arrival time, X_n, as $X_n = S_n - S_{n-1} \geq 0$. It follows that the sequence $\{S_n, n \geq 1\}$ can be rewritten in terms of the sequence $\{X_n, n \geq 1\}$ by using the relation $S_n = \sum_{i=1}^n X_i$. In the same way, the process $N(t)$ can be expressed in terms of the sequence $\{S_n, n \geq 1\}$ by the relation

$$N(t) = \max\{n : S_n \leq t\} \tag{1}$$

and we can reconstruct the sequence $\{S_n, n \geq 1\}$ by the function $N(t)$ via the reverse relation

$$S_n = \sup\{t : N(t) < n\} \tag{2}$$

It follows that we can characterize a homogeneous Poisson process by characterize one of the these three objects: the sequence of the arrival times, $\{S_n, n \geq 1\}$, the sequence of the inter-arrival times, $\{X_n, n \geq 1\}$, or the counting function $\{N(t), t \geq 0\}$.

We start by describing the distribution of the inter-arrival times, that according to the following proposition is given by a sequence of i.i.d. random variables.

Proposition 1. *Let $\{N(t), t \geq 0\}$ be a homogeneous Poisson process with rate λ, then the inter-arrival times $\{X_n, n \geq 1\}$ are independent random variables, each of them distributed as an exponential distribution with parameter λ.*

Proof. Using the fact that the event $\{X_1 > t\}$ is the same as the event $\{N(t) = 0\}$ we immediately get, using the fact that $N(t) \sim \text{Po}(\lambda\, t)$, that

$$\bar{F}_{X_1}(t) = \mathbb{P}\{X_1 > t\} = \mathbb{P}\{N(t) = 0\} = e^{-\lambda\, t} \; ,$$

that is the tail distribution of an exponential random variable with the parameter λ.

To get the tail distribution of X_2 we condition on the value of the first arrival time, $S_1 = X_1$, that is

$$\bar{F}_{X_2}(t) = \mathbb{P}\{X_2 > t\} = \int_0^\infty \mathbb{P}\{X_2 > t, X_1 \in ds\} = \int_0^\infty \mathbb{P}\{X_2 > t | X_1 = s\}\, \mathbb{P}\{X_1 \in ds\} \; . \tag{3}$$

The event $\{X_1 = s\}$ is the same as the event $\{N(s) = 1, N[0, s) = 0\}$, and in addition $\{X_2 > t\} \cap \{X_1 = s\} = \{N(s+t) = 1\} \cap \{X_1 = s\}$, hence

$$\begin{aligned} \mathbb{P}\{X_2 > t | X_1 = s\} &= \mathbb{P}\{N(s+t) = 1 | N(s) = 1,\, N[0, s) = 0\} \\ &= \mathbb{P}\{N(s+t) - N(s) = 0 | N(s) = 1,\, N(u) = 0, 0 \leq u < s\} \\ &\overset{\perp}{=} \mathbb{P}\{N(s+t) - N(s) = 0\} = e^{-\lambda\, t} \; . \end{aligned}$$

This implies that $X_1 \perp X_2$, and after substituting the expression above in equation (3) that $X_2 \sim \text{Exp}(\lambda)$ as well. Repeating the same argument for the other variables X_n, with $n > 2$, the result holds true. $\square$

Knowing the joint distribution of the fundamental sequence $\{X_n, n \geq 1\}$, it is easy to compute the marginal distribution for the sequence $\{S_n, n \geq 1\}$. Indeed, having that $S_n = \sum_{i=1}^n X_i$, it follows that the n-th arrival time has the same distribution as the sum of n i.i.d. $\mathrm{Exp}(\lambda)$ random variables, that is known as the $\mathrm{Erlang}(n, \lambda)$ distribution. The following proposition shows how to compute its density function.

Proposition 2. *Let $\{N(t), t \geq 0\}$ be a homogeneous Poisson process with rate λ, then the n-th arrival time, S_n has the following density function*

$$f_{S_n}(t) = \lambda \, \frac{(\lambda\, t)^{n-1}}{n-1!} \, e^{-\lambda t}$$

that corresponds to the density function of an Erlang(n, λ) random variable.

Proof. Take $n > 1$, having that $\{S_n \leq t\} \Leftrightarrow \{N(t) \geq n\}$, it immediately follows that

$$F_{S_n}(t) = \sum_{i=n}^{\infty} \frac{(\lambda\, t)^i}{i!} \, e^{-\lambda t} \ .$$

By differentiating with respect to t we have that

$$f_{S_n}(t) = \lambda e^{-\lambda t} \left(\sum_{i=n}^{\infty} \frac{(\lambda\, t)^{i-1}}{i-1!} - \sum_{i=n}^{\infty} \frac{(\lambda\, t)^i}{i!} \right) = \lambda e^{-\lambda t} \left(\sum_{i=n-1}^{\infty} \frac{(\lambda\, t)^i}{i!} - \sum_{i=n}^{\infty} \frac{(\lambda\, t)^i}{i!} \right) = \lambda e^{-\lambda t} \frac{(\lambda\, t)^{n-1}}{n-1!} \ .$$

The same result could be obtained heuristically by noticing that the event $\{S_n \in (t, t+dt)\}$ is the same as the event $\{N(t) = n-1, N(t, t+dt] = 1\}$ and therefore

$$\begin{aligned} f_{S_n}(t)\, dt &= \mathbb{P}\{S_n \in (t, t+dt)\} = \mathbb{P}\{N(t, t+dt] = 1 | N(t) = n-1, \} \, \mathbb{P}\{N(t) = n-1, \} \\ &\overset{\perp}{=} \mathbb{P}\{N(t, t+dt] = 1\} \, \mathbb{P}\{N(t) = n-1, \} = \lambda \, dt \, e^{-\lambda dt} \, \frac{(\lambda\, t)^{n-1}}{n-1!} \, e^{-\lambda\, t} \ . \end{aligned}$$

The result then follows by dividing both terms by dt and computing the limit as $dt \to 0$. □

Distribution of the points in a interval conditioned to the total number of points

In this section we study what is the joint distribution of the points of a homogeneous Poisson process in an interval given that we know already how many points fell in that interval.

Proposition 3. *Conditioned on the even $\{N(t) = n\}$ the random vector $(S_1, \ldots, S_n)$ has the same distribution of the ordered statistics $(U_{(1)}, \ldots, U_{(n)})$, where the U_i, $1 \leq i \leq n$, are independent random variables uniformly distributed in $[0, t]$.*

Proof. According to Proposition 5 we need to prove that

$$f_{S_1,\ldots,S_n|N(t)=n}(t_1, \ldots, t_n) = \frac{1_{\{0 \leq t_1 \leq \ldots \leq t_n \leq t\}}}{t^n}.$$

Let $t_{n+1} = t$, We start by computing the following probability,

$$\Delta F_{S_1,\ldots,S_n|N(t)=n}(t_1, \ldots, t_n) \overset{\Delta}{=} \mathbb{P}(S_i \in (t_i, t_i + dt_i), 1 \leq i \leq n | N(t) = n)$$

and then we compute the density in the following way

$$f_{S_1,\ldots,S_n|N(t)=n}(t_1, \ldots, t_n) = \lim_{dt_i \to 0,\, 1 \leq i \leq n} \frac{\Delta F_{S_1,\ldots,S_n|N(t)=n}(t_1, \ldots, t_n)}{\prod_{i=1}^n dt_i} \ . \tag{4}$$

We have that

$$\begin{aligned} \Delta F_{S_1,\ldots,S_n|N(t)=n}(t_1, \ldots, t_n) &= \frac{\mathbb{P}(S_i \in (t_i, t_i + dt_i), 1 \leq i \leq n, N(t) = n)}{\mathbb{P}(N(t) = n)} \\ &= \frac{\mathbb{P}(N(t_1) = 0,\, N(t_i, t_i + dt_i) = 1,\, N(t_i + dt_i, t_{i+1}) = 0, 1 \leq i \leq n)}{\mathbb{P}(N(t) = n)} \\ &\overset{\perp}{=} \frac{\mathbb{P}(N(t_1) = 0) \prod_{i=1}^n \mathbb{P}(N(t_i, t_i + dt_i) = 1) \, \mathbb{P}(N(t_i + dt_i, t_{i+1}) = 0)}{\mathbb{P}(N(t) = n)} \\ &= \frac{e^{-\lambda\, t_1} \prod_{i=1}^n (\lambda\, dt_i) e^{-\lambda\, dt_i} e^{-\lambda\,(t_{i+1} - t_i - dt_i)}}{(\lambda\, t)^n \, e^{-\lambda\, t}/n!} = \frac{n!}{t^n} \prod_{i=1}^n dt_i \end{aligned}$$

and therefore computing the limit in (4) we finally get

$$f_{S_1,\ldots,S_n|N(t)=n}(t_1, \ldots, t_n) = \frac{n!}{t^n} \ ,$$

that according to Proposition 5 corresponds to the joint density function of the order statistics of n independent uniform random variables in $[0, t]$. □

The Poisson process on the line

In this section we define the general Poisson process on the line, that is we do not assume that the intensity rate is the same at each time.

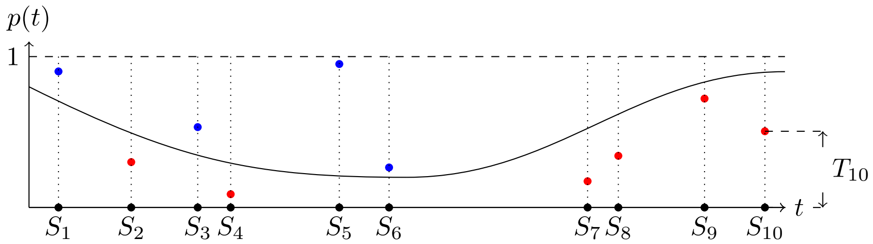
Definition 3. *A **Poisson process**, $N(t)$, with rate $\lambda(t)$ is defined as a counting process with independent increments with the property that the number of points counted in an interval $(t, t+s]$ is given by a Poisson distribution with parameter $\int_t^{t+s} \lambda(s)\, ds$, i.e.*

$$N(t, t+s] \sim Po(\Lambda[t, t+s]) \ ,$$

where Λ is defined as the intensity measure, with $\Lambda(A) = \int_A \lambda(t)\, dt$.

The Poisson process has some nice properties, that is we can make some operations with its points and get after the operation again a new Poisson process. One example is given in Exercise 2 where a new Poisson process is generated by joining two independent ones. In the next proposition we analyze the reverse case.

Proposition 4 (Splitting of a Poisson Process)**.** *Assume that you have a homogeneous Poisson process, $N(t)$, with rate λ, and that each time there is an arrival that point is colored in red or blue with a probability that depends on its arrival epoch and independently of the colors the other points were painted. For example, if S_n denotes the position of the n-th point, then it will be colored red with probability $p(S_n)$ and blue with probability $q(S_n) = 1 - p(S_n)$. Define $N^R(t)$ and $N^B(t)$, the processes that count respectively only the "red" and the "blue" points. Then $N^R(t)$ and $N^B(t)$ are non-homogeneous Poisson processes with intensity rates $\lambda^R(t) = \lambda\, p(t)$ and $\lambda^B(t) = \lambda\, q(t)$.*



Proof. Since the process $N(t)$ is a homogeneous Poisson process and the points counted by the processes $N_R(t)$ and $N_B(t)$ are the same, we immediately get that these last two processes are counting processes and that the number of points counted in disjoint intervals are independent. What is left to prove is that $N_R(t)$ and $N_B(t)$ are Poisson distributed, and in addition we prove that they are mutually independent, that is we show that

$$\mathbb{P}(N_R(t) = n,\, N_B(t) = m) = \frac{\Lambda_R(t)^n}{n!} e^{-\Lambda_R(t)}\, \frac{\Lambda_B(t)^m}{m!} e^{-\Lambda_B(t)} \ , \tag{5}$$

where $\Lambda_R(t) = \frac{\lambda}{t} \int_0^t p(s) ds$ and $\Lambda_B(t) = \frac{\lambda}{t} \int_0^t q(s) ds$, that gives the right intensity rates $\lambda_R(t) = \Lambda'_R(t) = \lambda\, p(t)$ and $\lambda_B(t) = \Lambda'_B(t) = \lambda\, q(t)$.

We prove by using conditioning, i.e.

$$\begin{aligned}
\mathbb{P}(N_R(t) = n,\, N_B(t) = m) &= \mathbb{P}(N_R(t) = n | N(t) = n+m)\, \mathbb{P}(N(t) = n+m) \\
&= \mathbb{P}(\sum_{i=1}^{n+m} 1\{T_i \le p(S_i)\} = n | N(t) = n+m)\, \frac{(\lambda\, t)^{n+m}}{n+m!} e^{-\lambda\, t} \\
&= \mathbb{P}(\sum_{i=1}^{n+m} 1\{T_i \le p(U_{(i)})\} = n)\, \frac{(\lambda\, t)^{n+m}}{n+m!} e^{-\lambda\, t} \\
&= \mathbb{P}(\sum_{i=1}^{n+m} 1\{T_i \le p(U_i)\} = n)\, \frac{(\lambda\, t)^{n+m}}{n+m!} e^{-\lambda\, t} \qquad (6)
\end{aligned}$$

where $\{T_i, 1 \le i \le n+m\}$ are i.i.d. U$(0,1)$ random variables such that when $T_i \le p(S_i)$ it means that we color the point S_i "red" and when $T_i \le p(S_i)$ it means that we color the point S_i "blue". In the third equality we used Proposition 3. The random variables $1\{T_i \le p(U_i)\}$ are i.i.d. Bernoulli with parameter

$$p = \int_0^t p(u)\, dF_{U_i}(u) = \frac{1}{t} \int_0^t p(u)\, du \ ,$$

so it follows that the random variable $Y = \sum_{i=1}^{n+m} 1\{T_i \le p(U_i)\} \sim \mathrm{Bin}(n+m,p)$. Substituting in (6) we finally get

$$\begin{aligned} \mathbb{P}(N_R(t)=n,\, N_B(t)=m) &= \mathbb{P}(Y=n)\,\frac{(\lambda\,t)^n}{n!}e^{-\lambda\,t} = \binom{n+m}{n} p^n\,q^m\,\frac{(\lambda\,t)^{n+m}}{n+m!}e^{-\lambda\,t} \\ &= \frac{n+m!}{n!\,m!}\,p^n\,q^m\,\frac{(\lambda\,t)^{n+m}}{n+m!}e^{-\lambda(p+q)\,t} \\ &= \frac{(\lambda\,p\,t)^n}{n!}\,e^{-\lambda\,p\,t}\,\frac{(\lambda\,q\,t)^m}{m!}\,e^{-\lambda\,q\,t} \end{aligned} \tag{7}$$

with $q = 1-p$ that is the same as equation (5). □

Exercises

1. Simulate a homogenous Poisson process with rate $\lambda > 0$, $N(\cdot)$, in a fixed interval $[0,t]$ using three different techniques:
   - Approximating it by a set of uniform random variables $\{U_i\}_i$ in a interval $[0,T]$ with $T >> t$.
   - Using a sequence of i.i.d. exponential random variables $\{X_i\}$ with parameter λ as inter-arrival times.
   - By using the Proposition 3, simulating first the total number of points falling in the interval $[0,t]$, i.e. $N(t)$, and then positioning the points uniformly in this interval.
2. Given two independent homogeneous Poisson processes, $N^I(t)$ and $N^{II}(t)$, with rates λ^I and λ^{II} respectively. Prove that the process $N(t) = N^I(t) + N^{II}(t)$ is again a Poisson process and find its rate.

A Appendix

A.1 Sum of independent random variables

Given two independent random variables X and Y, with distribution functions $F_X(x) = \mathbb{P}\{X \le x\}$ and $F_Y(y) = \mathbb{P}\{Y \le y\}$ if we define $Z = X+Y$, the distribution function $F_Z(z)$ is the given by

$$\begin{aligned} F_Z(z) = \mathbb{P}\{Z \le z\} = \mathbb{P}\{X+Y \le z\} &= \int_{-\infty}^{\infty} \mathbb{P}\{X+Y \le z | X=x\}\,dF_X(x) \\ &= \int_{-\infty}^{\infty} \mathbb{P}\{Y \le z-x | X=x\}\,dF_X(x) \\ &\overset{\perp}{=} \int_{-\infty}^{\infty} \mathbb{P}\{Y \le z-x\}\,dF_X(x) \\ &= \int_{-\infty}^{\infty} F_Y(z-x)\,dF_X(x) \end{aligned} \tag{8}$$

And if we define the Stieltjes convolution operator $F * G(t)$ that given two distribution functions $F(t)$, $G(t)$ is equal to

$$F * G(t) = \int_{-\infty}^{\infty} F(t-s)\,dG(s)$$

we get that

$$F_Z(z) = F_Y * F_X(z).$$

It is easy to check that $F * G(t) = G * F(t)$ by a simple change of variable or by simply noting that $Z = X+Y = Y+X$. If we differentiate the expression (8) and assuming that all the random variable have a density function, we have that

$$f_Z(z) = F'_z(z) = \int_{-\infty}^{\infty} f_Y(z-x)\,dF_X(x) = \int_{-\infty}^{\infty} f_Y(z-x)\,f_X(x)\,dx \tag{9}$$

and defining the convolution between two functions as $f * g(t) = \int_{-\infty}^{\infty} f(t-s)\,g(s)\,ds$ we have that

$$f_Z(z) = f_Y * f_X(z).$$

A.1.1 Positive random variables

Assuming now that $X, Y \geq 0$, we have that $F_X(x) = 0$ for $x < 0$ and also $F_Y(y) = 0$ for $y < 0$. Note that we admit $F_X(0) > 0$ and $F_Y(0) > 0$. In this case equation (8) becomes

$$F_Z(z) = \int_0^z F_Y(z-x)\, dF_X(x) = \int_0^z F_X(z-y)\, dF_Y(y) \tag{10}$$

still it is valid that $F_Z(z) = F_Y * F_X(z) = F_X * F_Y(z)$ that now reduces to the expression (10).

A.1.2 Laplace transforms

Assume that X has Laplace transform $\phi_X(s) = \tilde{F}_X(s) = \mathbb{E}[e^{s\,X}]$, and in the same way Y has Laplace transform $\phi_Y(s) = \tilde{F}_Y(s) = \mathbb{E}[e^{s\,Y}]$, then it follows that $Z = X + Y$ has Laplace transform $\phi_Z(s) = \tilde{F}_Z(s)$ given by

$$\begin{aligned}\tilde{F}_Z(s) &= \mathbb{E}[e^{s\,Z}] = \mathbb{E}[e^{s\,(X+Y)}] = \mathbb{E}[e^{s\,X}\, e^{s\,Y}] \\ &\overset{\perp}{=} \mathbb{E}[e^{s\,X}]\mathbb{E}[e^{s\,Y}] = \tilde{F}_X(s)\,\tilde{F}_Y(s)\end{aligned}$$

Last equation expresses the fact that the Laplace transform translates the convolution operator to a product operator, i.e.

$$\mathcal{L}(F * G)(s) = \mathcal{L}(F)(s)\,\mathcal{L}(G)(s),$$

where we denoted by $\mathcal{L}(F)(s)$ the Laplace-Stieltjes transform of the distribution function $F(t)$, i.e.

$$\mathcal{L}(F)(s) = \int_0^\infty e^{-s\,t}\, dF(t).$$

A.1.3 Sum of i.i.d. positive random variables

Assume to have two i.i.d. random variables X_1 and X_2 with common distribution function $F_X(x)$. Now $Y = X_1 + X_2$ has distribution function

$$F_Y(y) = F_X * F_X(y) = F_X^{*2}(y),$$

known also has 2-fold convolution of $F_X(x)$. In general if $Z = \sum_{i=1}^n X_i$ with X_i i.i.d. and with common distribution $F_X(x)$, then the distribution of Z is given by the n-th convolution of $F_X(x)$, i.e.

$$F_Z(z) = F_X^{*n}(z)$$

with $F_X^{*n}(z) = F_X^{*(n-1)} * F_X(z)$, that in the transformed domain corresponds to

$$\phi_Z(s) = \tilde{F}_Z(s) = \left(\tilde{F}_X(s)\right)^n.$$

A.2 Order Statisctics

Given a vector of n random variables $(Y_1, Y_2, \ldots, Y_n)$ we say that the corresponding vector of order statistics $(Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)})$ is the vector constructed by reordering the random variable in increasing order, that is

$$Y_{(1)} \leq Y_{(2)} \leq \ldots \leq Y_{(n)}\ .$$

If we assume that the vector of the staring random variables $(Y_i, 1 \leq i \leq n)$ is made of i.i.d. random variables with common density $f_Y(y)$ we can easily compute the joint density function by using the independence property, and getting the following result

$$f_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n) = \prod_{i=1}^n f_Y(y_i)\ .$$

Let $\Pi(y_1, \ldots, y_n)$ be the set of permutations of the vector $(y_1, \ldots, y_n)$, that is if $\pi \in \Pi(y_1, \ldots, y_n)$ is a given permutation then there is a bijective function $i(k)$ from the set $1 \leq k \leq n$ to itself, such that $\pi = (y_{i(1)}, \ldots, y_{i(n)}\}$, then we have that the number of elements of $\Pi(y_1, \ldots, y_n)$, is equal to $n!$. It follows that the joint density function of the statistic ordering vector $(Y_{(i)}, 1 \leq i \leq n)$ is given by

$$\begin{aligned} f_{Y_{(1)},\ldots,Y_{(n)}}(y_1, \ldots, y_n) &= 1_{\{y_1 \leq y_2 \leq \ldots \leq y_n\}} \sum_{\pi \in \Pi(y_1,\ldots,y_n)} \left(\prod_{i=1}^n f_Y(\pi_i)\right) \\ &= 1_{\{y_1 \leq y_2 \leq \ldots \leq y_n\}} \sum_{\pi \in \Pi(y_1,\ldots,y_n)} \left(\prod_{i=1}^n f_Y(y_i)\right) \\ &= 1_{\{y_1 \leq y_2 \leq \ldots \leq y_n\}}\, n! \prod_{i=1}^n f_Y(y_i)\ . \end{aligned}$$

In particular if we assume that the random vector $(Y_i, 1 \leq i \leq n)$ is made of i.i.d. uniform random variables in $[0, t]$ we have the following result.

Proposition 5. *Assuming that the vector $(U_i, 1 \leq i \leq n)$ is made of n independent random variables each uniformly distributed in $[0, t]$, then its joint density function is given by*

$$f_{U_1,\ldots,U_n}(u_1, \ldots, u_n) = \frac{\prod_{i=1}^{n} 1_{\{0 \leq u_i \leq t\}}}{t^n} \ ,$$

and the one of the corresponding order statistics is given by

$$f_{U_{(1)},\ldots,U_{(n)}}(u_1, \ldots, u_n) = \frac{1_{\{0 \leq u_1 \leq \ldots \leq u_n \leq t\}}}{t^n} \ .$$

A.3 Excercises

1. Assume X_1 and X_2 i.i.d uniformly distributed between $[0, 1]$, compute the distribution function of $Y = X_1 + X_2$. Compute its Laplace transform as well.
2. Assume X_n i.i.d exponential random variables with parameter $\lambda > 0$. Compute the distribution of $Z = \sum_{i=1}^{n} X_i$. Do you recognize this distribution? Compute its Laplace transform as well.
3. If $\Phi_(z)$ is the CDF of a standard Normal random variable. What would be its n-th fold convolution?