# Notes

**February $14^{\text{th}}$, 2012**

## The homogeneous Poisson process as limit of uniform distribution of independent points

The idea of this section is to construct the Poisson process as the limiting result of a sequence of stochastic processes. These consist in the experiment of throwing a number of points in a finite interval, in an independent and uniform way.

We define the process $N_T(t)$ in the following way. We assume that we have $n(T)$ points, let us say $\{X_i\}_i$ with $1 \leq i \leq n(T)$, and we distribute them independently and uniformly in the interval $[0, T]$. $N_T(t)$ then counts the number of points fallen in the interval $(0, t]$ and in general $N_T(s, t]$ counts the number of points that fell in the interval $(s, t]$ with $s \leq t$.

The idea is to let $T \to \infty$ and then to study the properties of the limiting process. In order to get a non degenerate limit, as soon as we take $T \to \infty$, we have to correspondingly increase the number of points we allocate in the interval.

Let assume that, in the end, we would like to have an average density of $\lambda$ points per time unit, it follows that

$$\lambda = \lim_{T\to\infty} \mathbb{E}[N_T(t, t+1]] \ . \tag{1}$$

Writing the expression in the expectation in the following way

$$N_T(t, t+1] = \sum_{i=1}^{n(T)} 1\{X_i \in (t, t+1]\} \ , \tag{2}$$

and substituting it in (1) we get

$$\begin{aligned} \lambda &= \lim_{T\to\infty} \mathbb{E}[\sum_{i=1}^{n(T)} 1\{X_i \in (t, t+1]\}] = \lim_{T\to\infty} n(T)\, \mathbb{E}[1\{X_i \in (t, t+1]\}] \\ &= \lim_{T\to\infty} n(T)\, \mathbb{P}(X_i \in (t, t+1]) = \lim_{T\to\infty} \frac{n(T)}{T} \ . \end{aligned}$$

From the relation above we see that in order to get the required limiting density of point the number of points to be thrown has to grow linearly with the size of the interval, i.e.

$$n(T) \approx \lfloor \lambda T \rfloor \ . \tag{3}$$

A natural question to address is to ask what would be the limit distribution of the number of points in a fixed interval $(s, t]$, with $s \leq t$. We have the following result.

**Proposition 1.** *The number of points in the interval $s, t]$ with $s \leq t$ will converge in distribution to a Poisson random variable with parameter $\lambda\,(t - s)$. That is*

$$N_T(s, t] \xrightarrow{\mathcal{L}} Po(\lambda(t - s)) \quad \text{as} \quad T \to \infty \tag{4}$$

*with $n(T) = \lambda T$.*

*Proof.* To prove the convergence in (4) we use the converge of characteristic functions. We define by $\phi_T(z) = \mathbb{E}[z^{N_T(s,t]}]$, with $0 < z < 1$, the characteristic function of the random variable $N_T(s, t]$ and we are going to prove that $\phi_T(z) \to \exp\{-\lambda(t - s)(1 - z)\}$.

Using again the expression $N_T(s, t] = \sum_{i=1}^{n(T)} 1\{X_i \in (s, t]\}$ we have that

$$\phi_T(z) = \mathbb{E}[z^{\sum_{i=1}^{n(T)} 1\{X_i \in (s,t]\}}] \stackrel{\perp}{=} \prod_{i=1}^{n(T)} \mathbb{E}[z^{1\{X_i \in (s,t]\}}] = \Big(\mathbb{E}[z^{1\{X_1 \in (s,t]\}}]\Big)^{n(T)} \ .$$

The generating function of the Bernoulli variable $1\{X_1 \in (s, t]\}$ is given by

$$\mathbb{E}[z^{1\{X_1 \in (s,t]\}}] = z \times \mathbb{P}(X_1 \in (s, t]) + 1 \times \mathbb{P}(X_1 \in (s, t]) = 1 - \frac{t - s}{T}(1 - z)$$

so that we can finally write

$$\phi_T(z) = \exp\{n(T)\, \ln(1 - \frac{t-s}{T}(1-z))\} \ .$$

Then the following steps show the required convergence

$$\begin{aligned}
\lim_{T\to\infty} \lfloor \lambda T \rfloor \ln(1 - \frac{t-s}{T}(1-z)) &= \lim_{T\to\infty} \frac{\lfloor \lambda T \rfloor}{\lambda T} \lim_{T\to\infty} \lambda T \ln(1 - \frac{t-s}{T}(1-z)) \\
&= \lambda \lim_{y\to 0} \frac{\ln(1 - y(t-s)(1-z))}{y} = \lambda \lim_{y\to 0} -\frac{(t-s)(1-z)}{1-y(t-s)(1-z)} \\
&= -\lambda(t-s)(1-z) \ ,
\end{aligned}$$

where in the second equality we used the substitution $y = 1/T$ and in the third one we used the De L'Hopital's rule. □

An another important property of the limit process is given by the independence of the number of points falling into two disjoint interval.

**Proposition 2.** *Given two disjoint intervals $(t_1, t_2]$ and $(t_3, t_4])$ with $t_1 < t_2 < t_3 < t_4$, then the following relation holds*

$$\lim_{T\to\infty} \mathbb{P}(N_T(t_1,t_2] = n, N_T(t_3,t_4] = m) = \lim_{T\to\infty} \mathbb{P}(N_T(t_1,t_2] = n)\, \mathbb{P}(N_T(t_3,t_4] = m) \tag{5}$$

*showing that in the limit the number of points falling into two disjoint intervals are independent.*

*Proof.* We can write the term in the limit in the left end side of equation (5) in the following way

$$\mathbb{P}(N_T(t_1,t_2] = n, N_T(t_3,t_4] = m) = \mathbb{P}(N_T(t_1,t_2] = n | N_T(I) = n+m)\, \mathbb{P}(N_T(I) = n+m)$$

where we have defined the set $I = (t_1, t_2] \cup (t_3, t_4]$. It follows that the random variable $N_T(t_1, t_2]$ conditioned on $N_T(I) = n + m$ is distributes as a Binomial distribution with parameters $(n + m, p)$ where $p$ is the probability that any of the random points falling in the interval $I$ will actually fall in the subinterval $(t_1, t_2]$, i.e. $p = (t_2 - t_1)/((t_4 - t_3) + (t_2 - t_1))$.

We have that

$$\mathbb{P}(N_T(t_1,t_2] = n | N_T(I) = n+m) = \binom{n+m}{n} \left( \frac{t_2 - t_1}{(t_4-t_3)+(t_2-t_1)} \right)^n \left( \frac{t_4 - t_3}{(t_4-t_3)+(t_2-t_1)} \right)^m$$

and by Proposition 1

$$\lim_{T\to\infty} \mathbb{P}(N_T(I) = n+m) = \frac{((t_4-t_3)+(t_2-t_1))^{n+m}}{n+m!} \lambda^n e^{-\lambda(t_2-t_1)}\, \lambda^m e^{-\lambda(t_4-t_3)}$$

whose product equals the right end side of equation (5) that by Proposition 1 is given by

$$\lim_{T\to\infty} \mathbb{P}(N_T(t_1,t_2] = n)\, \mathbb{P}(N_T(t_3,t_4] = m) = \frac{(\lambda\, (t_2-t_1))^n}{n!} e^{-\lambda(t_2-t_1)}\, \frac{(\lambda\, (t_4-t_3))^m}{m!} e^{-\lambda(t_4-t_3)} \ .$$

□

# Appendix

In this section we show some useful result.

**Lemma 1.** *Let $X$ be a Poisson random variable with parameter $\lambda$ then its generating function isn given by $\phi_X(z) = \exp\{-\lambda(1-z)\}$.*

*Proof.* The probability mass function of $X$ is given by $p_X(n) = \lambda^n/n!\, e^{-\lambda}$. It follows that

$$\phi_X(z) = \mathbb{E}[z^X] = \sum_{n=0}^{\infty} z^n\, p_X(n) = \sum_{n=0}^{\infty} \frac{(z\,\lambda)^n}{n!} e^{-\lambda} = e^{-\lambda(1-z)} \ .$$

□