Black and Scholes formula

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Additional Material

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The Black and Scholes

The *Black and Scholes* model assumes

- The interest rate $r > 0$ is constant. That is the discount factor is
$$R_t = \exp\{-\int_0^t r\, ds\} \ .$$
- There is only one risky asset
$$dS_t = S_t(\mu\, dt + \sigma\, dB_t) \ ,$$
with $\mu \in \mathbb{R}$ and $\sigma > 0$ constants, and B_t a $\mathbb{P}$-Brownian motion.

Market completeness

Proposition ([1] Prop. 2.3.1.2)

On the Black and Scholes model, there exists a unique e.m.m. $\mathbb{Q}$, precisely

$$\mathbb{Q}|_{\mathcal{F}_t} = \exp\{-\theta B_t - \frac{1}{2}\theta^2 t\}\, \mathbb{P}|_{\mathcal{F}_t}$$

where $\theta = (\mu - \sigma)/\sigma$ is the risk-premium.

The risk-neutral dynamics of the asset are

$$dS_t = S_t(r\, dt + \sigma dW_t)$$

where W is a $\mathbb{Q}$-Brownian motion.

$$S_t = S_0 \exp\{\mu t + \sigma B_t - \frac{\sigma^2}{2} t\} = S_0 e^{rt} \exp\{\sigma W_t - \frac{\sigma^2}{2} t\} = S_0 e^{\sigma X_t}$$

with $X_t = \nu t + W_t$ and $\nu = \frac{r}{\sigma} - \frac{\sigma}{2}$.

Pricing contingent claims

Proposition ([1] Prop. 2.3.1.3)

Let us consider the Black and Scholes framework

$$dS_t = S_t(r\,dt + \sigma dW_t), \quad S_0 = x$$

where W is a is a $\mathbb{Q}$-Brownian motion. and $\mathbb{Q}$ is the unique e.m.m. or risk-neutral probability. Then, the value of the contingent claim $h(S_T)$ is

$$\mathbb{E}_{\mathbb{Q}}[e^{-rT}h(S_T)] = e^{-\left(r+\frac{\nu^2}{2}\right)T}\mathbf{W}(e^{\nu X_T}h(x\,e^{\sigma X_T}))$$

where $\nu = \frac{r}{\sigma} - \frac{\sigma}{2}$ and $\mathbf{W}$ is the law of the Brownian motion on the canonical space with X being the canonical process.

European Call Option

Proposition ([1] Prop. 2.3.1.3)

For the Black and Scholes framework, the value at time t of a European call with maturity T and strike K — that is, the contingent claim $H = h(S_T) = (S_T - K)^+$ — is $\mathcal{BS}(S_t, \sigma, t)$, where

$$\begin{aligned}\mathcal{BS}(x, \sigma, t) =& x\,\mathcal{N}\left[d_1\left(\frac{x}{K\,e^{-r(T-t)}}, T-t\right)\right] \\ &- K\,e^{-r(T-t)}\mathcal{N}\left[d_2\left(\frac{x}{K\,e^{-r(T-t)}}, T-t\right)\right]\end{aligned}$$

where $\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}\,dy$ *and*

$$d_1(y,u) = \frac{1}{\sqrt{\sigma^2 u}}\ln(y) + \frac{1}{2}\sqrt{\sigma^2 u} \text{ and } d_2(y,u) = d_1(y,u) - \sqrt{\sigma^2 u}\ .$$

Bibliography

Bibliography

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