Enlargement of filtration

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Enlargement of Filtration ([1] 5.9)

If $\mathbb{G}$ is a filtration larger than $\mathbb{F}$, it is not true that an $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale.

> K Itö [3] was the first to look at problems of enlargement of filtrations.
>
> From the seventies, Barlow, Jeulin and Yor started a systematic study of the problem of enlargement of filtrations:

Which $\mathbb{F}$-martingale M remains a $\mathbb{G}$-semi-martingale, and if it is the case, what is the semi-martingale decomposition of M in $\mathbb{G}$?

Immersion of Filtration

Immersion of Filtration ([1] 5.9.1)

Let $\mathbb{F}$ and $\mathbb{G}$ be two filtrations such that $\mathbb{F} \subset \mathbb{G}$

When all $\mathbb{F}$-martingales are $\mathbb{G}$-martingales?

That is is equivalent to ask

$$\mathbb{E}[D|\mathcal{F}_t] = \mathbb{E}[D|\mathcal{G}_t], \forall t \text{ and } D \in L^1(\mathcal{F}_\infty) \ .$$

Just consider $\mathbb{G} = \mathbb{F} \vee \sigma(D)$ with $D \in L^1(\mathcal{F}_\infty)$ and D not $\mathcal{F}_0$ measurable.

$\mathbb{E}[D|\mathcal{G}_t] = D$ is a $\mathbb{G}$-martingale
$\mathbb{E}[D|\mathcal{F}_t]$ is an $\mathbb{F}$-martingale

However for $t = 0$, $\mathbb{E}[D|\mathcal{F}_t] \neq \mathbb{E}[D|\mathcal{G}_t] = D$, that implies that the $\mathbb{F}$-martingale $(\mathbb{E}[D|\mathcal{F}_t], t \geq 0)$ is not a $\mathbb{G}$-martingale.

Immersion of Filtration ([1] 5.9.1)

Definition

The filtration $\mathbb{F}$ is said to be *immersed* in $\mathbb{G}$ if any square integrable $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale.

this is also known as the $(\mathcal{H})$ hypothesis:

$(\mathcal{H})$ *Every $\mathbb{F}$-square integrable martingale is a $\mathbb{G}$-square integrable martingale*.

Proposition

Hypothesis $(\mathcal{H})$ is equivalent to any of the following properties:

$(\mathcal{H}1)$ $\forall t \geq 0$, *the σ-fields $\mathcal{F}_\infty$ and $\mathcal{G}_t$ are conditionally independent given $\mathcal{F}_t$*

$(\mathcal{H}2)$ $\forall t \geq 0, \forall G_t \in L^1(\mathcal{G}_t), \mathbb{E}[G_t|\mathcal{F}_\infty] = \mathbb{E}[G_t|\mathcal{F}_t]$

$(\mathcal{H}3)$ $\forall t \geq 0, \forall F \in L^1(\mathcal{F}_\infty), \mathbb{E}[F|\mathcal{G}_t] = \mathbb{E}[F|\mathcal{F}_t]$

In particular, $(\mathcal{H})$ holds if and only if every $\mathbb{F}$-local martingale is a $\mathbb{G}$-local martingale

The Brownian Bridge as example of Initial Enlargement

The Brownian Bridge example ([1] 5.9.2)

Let $B = (B_t, t \geq 0)$ be an $\mathbb{F}^B$-Brownian motion, and

$$\mathcal{G}_t = \mathcal{F}_t^B \vee \sigma(B_1) \ .$$

In $\mathbb{G}$ the process B is not anymore a martingale, since

$$\mathbb{E}[B_t | \mathcal{G}_t] = B_1 \neq B_t \ .$$

Actually as we will see later b is a $\mathbb{G}$-semi-martingale with decomposition

$$B_t = \beta_t + \int_0^{t \wedge 1} \frac{B_1 - B_s}{1 - s} ds \ ,$$

where β is a $\mathbb{G}$-Brownian motion.

The Brownian Bridge ([1] 4.3.5)

Definition

The *Brownian Bridge*, $(b_t, 0 \leq t \leq 1)$ is defined as the conditioned process

$$(B_t, t \leq 1 | B_1 = 0) \ .$$

The Brownian Bridge is a gaussian process, and each of the processes W, X, Y and Z below are examples of Brownian Bridges

$$W_t = B_t - t\, B_1, \quad 0 \leq t \leq 1$$

$$X_t = (1-t) \int_0^t \frac{dB_s}{1-s}, \quad 0 \leq t \leq 1$$

$$Y_t = (1-t)\, B\left(\frac{t}{1-t}\right), \quad 0 \leq t \leq 1$$

$$Z_t = t\, B\left(\frac{1-t}{t}\right), \quad 0 \leq t \leq 1$$

The laws on the canonical space

Definition

We denote by $\mathbf{W}^{(1)}_{0->1}$ the law of the Brownian Bridge on the *canonical space*.

Definition

We denote by $\mathbf{W}^{(T)}_{x->y}$ the law of the Brownian Bridge between x and y during the time interval $[0, T]$, on the *canonical space*. That is the process

$$\left(x + B_t - \frac{t}{T}B_T + \frac{t}{T}(y - x); 0 \leq t \leq T\right) .$$

Theorem ([1] Th. 4.3.5.1)

For every t, $\mathbf{W}^{(t)}_{x->y}$ is equivalent to $\mathbf{W}_x$ on $\mathcal{F}_s$ for $s < t$.

$\mathbf{W}_x$ is the Wiener measure such that $\mathbf{W}_x(X_0 = x) = 1$.

Markovian Bridges

For general Markov processes with semi-group

$$P_t(x, dy) = p_t(x, y)\, dy \ ,$$

$$\boxed{\mathbb{P}^{(t)}_{x->y}|_{\mathcal{F}_s} = \frac{p_{t-s}(X_s, y)}{p_t(x, y)} \mathbb{P}_x|_{\mathcal{F}_s}}$$

n-dimensional Brownian motion

For $x = y = 0$ and $s < t$

$$\boxed{\mathbf{W}^{(t)}_{0->0}|_{\mathcal{F}_s} = \left(\frac{t}{t-s}\right)^{n/2} \exp\left(\frac{-|X_s|^2}{2(t-s)}\right) \mathbf{W}_0|_{\mathcal{F}_s}}$$

Identifying the density as the exponential martingale $\mathcal{E}(Z)$, where

$$Z_s = -\int_0^s \frac{X_u}{t-u} dX_u \ ,$$

the canonical decomposition of the standard Brownian bridge (under $\mathbf{W}^{(t)}_{0->0}$) is:

$$X_s = \beta_s - \int_0^s \frac{X_u}{t-u}\, du, \quad s < t \ ,$$

where $(\beta_s, s \leq t)$ is a Brownian motion under $\mathbf{W}^{(t)}_{0->0}$.

SDE for the Brownian bridge

The Brownian bridge is solution of the following SDE

$$\begin{cases} db_t &= -\frac{b_t}{1-t}dt + d\beta_t, \quad 0 \leq t < 1 \\ b_0 &= 0 \end{cases}$$

The Brownian Bridge example ([1] 5.9.2)

Proposition

The decomposition of B in the filtration $\mathbb{G}$ is

$$B_t = W_t + \int_0^{t\wedge 1} \frac{B_1 - B_s}{1-s} ds \ ,$$

where W is a $\mathbb{G}$-Brownian motion.

It follows that if M is an $\mathbb{F}$-local martingale such that $\int_0^1 \frac{1}{\sqrt{1-s}} d|\langle M, B\rangle|_s$ is finite, then

$$M_t = \tilde{M}_t + \int_0^{t\wedge 1} \frac{B_1 - B_s}{1-s} d\langle M, B\rangle_s \ ,$$

where $\tilde{M}$ is an $\mathbb{G}$-local martingale.

The Brownian Bridge example ([1] 5.9.2) i

Comment ([1] 5.9.2.2)

The singularity of $\xi_t = \frac{B_1 - B_t}{1-t}$ at $t = 1$, implies it is not square-integrable between 0 and 1.

This prevents Girsanov measure change to transform the $(\mathbb{G}, \mathbb{P})$ semi-martingale $(\mathbb{G}, \mathbb{Q})$-martingale.

Let $dS_t = S_t(\mu dt + \sigma\, dB_t)$ and enlarge the filtration with S_1 (or equivalently with B_1). In the enlarged filtration, the dynamics are

$$dS_t = S_t((\mu + \sigma\xi_t)dt + \sigma\, d\beta_t)\ ,$$

and there *does not exists* an e.m.m. such that the discounted price process $(e^{-rt}S_t, t \leq 1)$ s a $\mathbb{G}$-martingale.

The Brownian Bridge example ([1] 5.9.2) ii

However for every $\epsilon \in]0,1]$, there exists a uniformly integrable $\mathbb{G}$-martingale L defined as

$$dL_t = \frac{\mu - r - \sigma\,\xi_t}{\sigma} L_t\, d\beta_t, \quad t \leq 1-\epsilon, \quad L_0 = 1,$$

such that $d\mathbb{Q}|_{\mathcal{G}_t} = L_t\, \mathbb{P}|_{\mathcal{G}_t}$, the process $(e^{-rt} S_t, t \leq 1-\epsilon)$ is a $(\mathbb{G}, \mathbb{Q})$-martingale.

The information of the insider trader is too strong, such that it creates an arbitrage opportunity.

Bibliography

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