Basic Concepts and Examples in Finance

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July 5, 2017

ICMAT / UC3M

The Financial Market

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We assume there are

- d *risky assets* or *securities*: $S^i, i = 1, \ldots, d$
  They are assumed to be semi-martingales with respect to a filtration $\mathbb{F}$.
- One *riskless asset* or *saving account*: S^0
  Its dynamics are given by

$$dS_t^0 = S_t^0 \, r_t \, dt, \quad S_0^0 = 1 \ .$$

- r is the (positive) *interest rate*, assumed $\mathbb{F}$-adapted.

Discount factor

One monetary unit invested at time 0 in the riskless asset will give a *payoff* of $\exp\{\int_0^t r_s\, ds\}$ at time $t > 0$.

If r is deterministic, the price at time 0 of one monetary unit delivered at time t is $R_t = \exp\{-\int_0^t r_s\, ds\}$.

In general

$$R_t = (S_t^0)^{-1}$$

is called the *discount factor*.

Zero-Coupon bond (ZC)

The asset that delivers one monetary unit at time T is called a *zero-coupon bond* (ZC) of maturity T.

If r is deterministic, its price at time t is given by $P(t,T) = \exp(-\int_t^T r_s\, ds)$ and follows the dynamics

$$d_t P(t,T) = r_t\, P(t,T)\, dt, \quad P(T,T) = 1 \ .$$

In general the formula above is absurd for r stochastic (the left side is $\mathcal{F}_t$-measurable, while the right side is not), and

$$P(t,T) = (R_t)^{-1}\mathbb{E}_{\mathbb{Q}}[R_T|\mathcal{F}_t]$$

where $\mathbb{Q}$ is the risk probability measure.

Portfolio

A *portfolio* (or a *strategy*) is a $(d+1)$-dimensional $\mathbb{F}$-predictable process $\hat{\pi}$:

$$(\hat{\pi}_t = (\pi_t^i, i = 0, \ldots, d) = (\pi_t^0, \pi_t), \quad t \geq 0)$$

where π_t^i represents the number of shares of the asset i held at time t.

The *time-t value* of the portfolio $\hat{\pi}$ is given by

$$V_t(\hat{\pi}) = \sum_{i=0}^{d} \pi_t^i \, S_t^i = \pi_t^0 \, S_t^0 + \sum_{i=1}^{d} \pi_t^i \, S_t^i \ .$$

Some Assumptions

- borrowing and lending interest rates are equal to $(r_t, t \geq 0)$.
- there a no transaction costs (market liquidity)
- the number of shares of the asset available in the market is unbounded
- it is allowed *short-selling* ($\pi_t^i < 0$ for $i > 0$) as well as borrowing money ($\pi_t^0 < 0$)

Then we add the *self-financing* condition, that is changes in the value of portfolio are not due to rebalancing but only to changes in the asset prices. In continuous time it is a constraint and it is not a consequence of the *Itô lemma*.

Self-financing condition

Definition

A portfolio $\hat{\pi}$ is said to be *self-financing* if

$$dV_t(\hat{\pi}) = \sum_{i=0}^{d} \pi_t^i \, dS_t^i \ ,$$

or in an integral form,

$$V_t(\hat{\pi}) = V_0(\hat{\pi}) + \sum_{i=0}^{d} \int_0^t \pi_s^i \, dS_s^i \ .$$

We are going to assume that $\int_0^t \pi_s^i \, dS_s^i$ are well defined.

If $\hat{\pi} = (\pi^0, \pi)$ is a self-financing portfolio then

$$dV_t(\hat{\pi}) = r_t \ V_t(\hat{\pi}) \, dt + \pi_t \cdot (dS_t - r_t S_t \, dt) \ .$$

Self-financing condition

The self-financing condition holds also for the discounted processes

Proposition ([1] 2.1.1.3)

If $\hat{\pi} = (\pi^0, \pi)$ *is a self-financing portfolio then*

$$R_t\, V_t(\hat{\pi}) = V_0(\hat{\pi}) + \sum_{i=1}^{d} \int_0^t \pi_s^i \, d(R_s\, S_s^i) \ ,$$

or equivalently

$$dV_t^0(\hat{\pi}) = \sum_{i=1}^{d} \pi_t^i \, dS_t^{i,0} \ ,$$

where $V_t^0 = V_t\, R_t = V_t / S_t^0$ *and* $S_t^{i,0} = S_t^i\, R_t = S_t^{i,0} / S_t^0$.

By abuse of language, we call $\pi = (\pi^1, \ldots, \pi^d)$ a self-financing portfolio.

Self-financing condition

Proposition ([1] 2.1.1.3 continue)

Conversely, if x *is a given positive real number,* $\pi = (\pi^1, \dots, \pi^d)$ *is a vector of predictable processes, and* V^π *denotes the solution of*

$$dV_t^\pi = r_t \; V_t^\pi \, dt + \pi_t \cdot (dS_t - r_t S_t \, dt), \quad V_0^\pi = x \; ,$$

then the $\mathbb{R}^{d+1}$*-valued process* $\hat{\pi} = (R(V^\pi - \pi \, S), \pi)$ *is a self financing strategy, and* $V_t^\pi = V_t(\hat{\pi})$*.*

Exercise (2.1.1.4)

Let $dS_t = (\mu \, dt + \sigma \, dW_t)$ *and* $r = 0$*.*

Is the portfolio $\hat{\pi} = (t, 1)$ *self-financing?*

If not, find π_t^0 *such that* $\hat{\pi} = (\pi_t^0, 1)$ *is self-financing.*

Arbritrage Opportunities

Arbitrage Opportunities ([1] 2.1.2)

An *arbitrage opportunity* is informally a self-financing strategy with 0 initial value and with terminal value $V_T(\hat{\pi}) \geq 0$, such that $\mathbb{E}[V_T(\hat{\pi})] > 0$.

Theorem (Dudley (1977))

Let X be an $\mathcal{F}_T^W$-random variable, then there exists a predictable process θ such that $\int_0^T \theta_s^2 < \infty$, a.s., and

$$X = \int_0^T \theta_s dW_s \ .$$

With $d = 1$, and $dS_s = \sigma S_s \, dW_s$, and $r = 0$, set $\pi_t = \theta_t / (\sigma S_t)$ with $\int_0^T \theta_s dW_s = A$, $A > 0$.

Equivalent Martingale Measure ([1] 2.1.3)

Definition

An *equivalent martingale measure* (*e.m.m.*) is a probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$ on $\mathcal{F}_T$, such that the discounted prices $(R_t S_t^i, t \leq T)$ are $\mathbb{Q}$-local martingales.

Folk Theorem: Protter 2001

Let S be the stock price process. There is absence of arbitrage *essentially* if and only if there exists a probability $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that the discounted price process is a $\mathbb{Q}$-local martingale.

Proposition

Under any e.m.m. the discounted value of a self-financing strategy is a local martingale.

Admissible strategies ([1] 2.1.4)

Definition

A self-financing strategy π is said to be admissible if there exists a constant A such that $V_t^\pi \geq -A$, a.s. for every $t \leq T$.

Definition

An arbitrage opportunity on the time interval $[0, T]$ is an admissible self-financing strategy π such that $V_0^\pi = 0$, $V_T^\pi \geq 0$ and $\mathbb{E}[V_T^\pi] > 0$.

Admissible strategies ([1] 2.1.4)

Following Delban and Schachermayer (1994)

$$\mathcal{K} = \{\int_0^T \pi_s \, dS_s : \pi \text{ is admissible}\}$$

$$\mathcal{A}_0 = \mathcal{K} - L^0_+ = \{\int_0^T \pi_s \, dS_s - f : \pi \text{ is admissible}, f \geq 0, f \text{ finite}\}$$

$$\mathcal{A} = \mathcal{A}_0 \cap L^\infty$$

$$\bar{\mathcal{A}} = \text{ closure of } \mathcal{A} \text{ in } L^\infty \ .$$

Definition

A semi-martingale S satisfies the *no-arbitrage condition* if $\mathcal{K} \cap L^\infty_+ = 0$. A semi-martingale S satisfies the *No-Free Lunch with Vanishing RIsk* (*NFLVR*) condition if $\bar{\mathcal{A}} \cap L^\infty_+ = 0$.

Admissible strategies ([1] 2.1.4)

Following Delban and Schachermayer (1994)

Theorem (Fundamental Theorem. See [1] Th. 2.1.4.4)
Let S be a locally bounded semi-martingale. There exists an equivalent measure $\mathbb{Q}$ for S if and only if S satisfies NFLVR.

Theorem (Th. 9.7.2 in [4])
Let S be an adapted cádlág process. If S is locally bounded and satisfies the NFLVR condition for simple integrands, then S is a semi-martingale.

Complete Market

Contingent claims and replicating strategies ([1] 2.1.5)

Definition

A *contingent claim*, H, is defined as a square integrable $\mathcal{F}_T$-random variable, where T is a fixed horizon.

Definition

A contingent claim H is said to be *edgeable* if there exists a predictable process $\pi = (\pi^1, \ldots, \pi^d)$ such that $V_T^\pi = H$. The self financing strategy $\hat{\pi} = (R(V^\pi - \pi S), \pi)$ is called *replicating strategy* (or the *hedging strategy*) of H, and $V_0(\pi) = h$ is the initial price. The process V^π is the price process of H.

Completeness

Definition

Assume that r is deterministic and let $\mathbb{F}^S$ be the natural filtration of the prices. The market is said to be *complete* if any contingent claim $H \in L^2(\mathcal{F}_T^S)$ is the value at time T of some self-financing strategy π.

Completeness

Theorem ([1] Th. 2.1.5.4)
Let $\tilde{S}$ be a process which represents the discounted prices. If there exists a unique e.m.m. $\mathbb{Q}$ such that $\tilde{S}$ is a $\mathbb{Q}$-local martingale, then the market is complete and arbitrage free.

Theorem ([1] Th. 2.1.5.5)
In an arbitrage free and complete market, the time-t price of a (bounded) contingent claim H is

$$V_t^H = R_t^{-1}\mathbb{E}_{\mathbb{Q}}[R_T\, H|\mathcal{F}_t]\ .$$

where $\mathbb{Q}$ is the unique e.m.m. and R is the discount factor.

Bibliography

Bibliography

- M. Jeanblanc, M. Yor and M. Chesney (2009). *Mathematical methods for financial markets*. Springer, London.
- D. Revuz and M. Yor (1999). *Continuous Martingales and Brownian Motion.* Springer Verlag, Berlin. 3rd ed.
- Ph. Protter (2001). *A partial introduction to financial asset pricing theory.* Stochastic Processes and their Appl., 91:169–204.
- F. Delban and W. Schachermayer (1994). *A general version of the fundamental theorem of asset pricing.* Math. Annal., 300:463–520.