Girsanov's Theorem

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Decomposition of $\mathbb{P}$-Martingales as $\mathbb{Q}$-semi-martingales

Theorem ([1] 1.7.2.1)

Let $\mathbb{P}$ and $\mathbb{Q}$ be locally equivalent with a continuous Radon-Nikodým density L. If M is a continuous $\mathbb{P}$-local martingale, then the process $\tilde{M}$ defined by

$$d\tilde{M} = dM - \frac{1}{L} d\langle M, L \rangle$$

is a continuous $\mathbb{Q}$-local martingale. If N is another continuous $\mathbb{P}$-local martingale $\langle M, N \rangle = \langle \tilde{M}, \tilde{N} \rangle = \langle M, \tilde{N} \rangle$.

Corollary ([1] 1.7.2.1)

We may write the process L as a Doléans-Dade martingale: $L_t = \mathcal{E}(\xi)$, where ξ is an $\mathbb{F}$-local martingale. The process $\tilde{M} = M - \langle M, \xi \rangle$ is a $\mathbb{Q}$-local martingale.

Girsanov's Threorem: One dimensional Brownian Motion

Proposition (Girsanov's Threorem, [1] 1.7.3.1)

Let W be a $(\mathbb{P}, \mathbb{F})$-Brownian motion and let θ be an $\mathbb{F}$-adapted process such that the solution of the SDE

$$dL_t = -L_t \theta_t \, dW_t, \quad L_0 = 1$$

is a martingale. We set $\mathbb{Q}|_{\mathcal{F}_t} = L_t \, \mathbb{P}|_{\mathcal{F}_t}$. Then the process W admits a $\mathbb{Q}$-semi-martingale decomposition $\tilde{W}$ as

$$W_t = \tilde{W}_t - \int_0^t \theta_s \, ds$$

where $\tilde{W}$ is a $\mathbb{Q}$-Brownian motion.

Girsanov's Threorem: One dimensional Brownian Motion

Example ([1] 1.7.3.3)

Consider the geometric Brownian motion, S, that is

$$dS_t = S_t(\mu\, dt + \sigma\, dW_t)\ .$$

Example ([1] 1.7.3.5)

Consider the process X solution of

$$dX_t = a\, dt + 2\sqrt{X_t}\, dW_t\ .$$

Exercises

Exercise ([1] 1.7.3.6)

Consider $\mathbb{Q} = h(W_T)\,\mathbb{P}$ *with* $h \in C_c^1(\mathbb{R}^+)$. *Prove that* B *defined by*

$$dB_t = dW_t - \left(\frac{\int_{-\infty}^{+\infty} \exp\{-(y - W_t)^2/(2(T-t))\}\, h'(y)\, dy}{\int_{-\infty}^{+\infty} \exp\{-(y - W_t)^2/(2(T-t))\}\, h(y)\, dy} \right) dt$$

is a $\mathbb{Q}$*-Brownian motion. See [3] for applications to finance.*

Exercise ([1] 1.7.3.7)

a) *Let* $dS_t = S_t\,\sigma\,dW_t$, $S_0 = x$. *Prove that for any bounded function* f

$$\mathbb{E}[f(S_T)) = \mathbb{E}\left[\frac{S_T}{x}\,f\left(\frac{x^2}{S_T}\right)\right]$$

b) *Prove that, if* $dS_t = S_t(\mu\,dt + \sigma\,dW_t)$, *there exists* γ *sich that* S^γ *is a martingale. Prove that for any bounded function* f

$$\mathbb{E}[f(S_T)) = \mathbb{E}\left[\left(\frac{S_T}{x}\right)^\gamma f\left(\frac{x^2}{S_T}\right)\right]$$

and

$$\mathbb{E}[S_T^\alpha f(S_T)) = x^\alpha e^{\alpha\,\mu\,T}\mathbb{E}\left[f\left(e^{\alpha\,\sigma^2\,T}S_T\right)\right] \quad .$$

Exercises

Exercise ([1] 1.7.3.8)

Let W be a $\mathbb{P}$-Brownian motion, and $B_t = W_t + \nu\, t$ be a $\mathbb{Q}$-Brownian motion, under a suitable change of probability. Check that, inthe case $\nu > 0$, the process $\exp\{W_t\}$ tend towards 0 under $\mathbb{Q}$ when t goes to ∞, whereas this is not the case under $\mathbb{P}$.

Exercise ([1] 1.7.3.10)

Let W be a $\mathbb{P}$-Brownian motion, and set $d\mathbb{Q}|_{\mathcal{F}_t} = L_t\, d\mathbb{P}|_{\mathcal{F}_t}$ where $L_t = \exp\{\lambda W_t - \frac{1}{2}\lambda^2 t\}$. Prove that the process X, where

$$X_t = W_t - \int_0^t \frac{W_s}{s}\, ds$$

is a Brownian motion with respect to its natuarl filtration under both $\mathbb{P}$ and $\mathbb{Q}$.

Girsanov's Threorem: Multi-dimensional Brownian Motion

Proposition (Girsanov's Threorem, [1] §1.7.4)

Let W be a $(\mathbb{P}, \mathbb{F})$-Brownian motion and let θ be an n-dimensional $\mathbb{F}$-adapted process such that $\int_0^t ||\theta_s||^2 ds < \infty$, a.s. Define the local martingale L as the solution of $dL_t = -L_t\, \theta_t \cdot dW_t = -L_t \left(\sum_i \theta_t^i\, dW_t^i\right)$, and $L_0 = 1$, so that

$$L_t = \exp\{-\int_0^t \theta_s \cdot dW_s - \frac{1}{2}\int_0^t ||\theta_s||^2\, ds\}\ .$$

If L is a martingale the n-dimensional process

$$(\tilde{W}_t = W_t + \int_0^t \theta_s\, ds, t \geq 0)$$

is a $\mathbb{Q}$-Brownian motion, where $\mathbb{Q}|_{\mathcal{F}_t} = L_t\, \mathbb{P}|_{\mathcal{F}_t}$.

If W is a Brownian motion with correlation matrix Λ, then $(\tilde{W}_t = W_t + \int_0^t \theta_s \cdot \Lambda\, ds, t \geq 0)$ is a $\mathbb{Q}$-Brownian motion with the same correlation matrix Λ.

PRP under Change of Probability ([1] 1.7.7)

Proposition

Let W be a $(\mathbb{P}, \mathbb{F}^W)$-Brownian motion and $\mathbb{Q}$ a probability measure locally equivalent to $\mathbb{P}$. Let $\tilde{W}$ be the martingale part of the $\mathbb{Q}$-semi-martingale.

If M is a $(\mathbb{Q}, \mathbb{F}^W)$-local martingale, there exists an $\mathbb{F}^W$-predictable process H such that

$$\forall t, \quad M_t = M_0 + \int_0^t H_s \, d\tilde{W}_s \ .$$

Invariance of BM under Change of Measure ([1] 1.7.8)

Proposition

Let X be a real valued $\mathbb{F}$-Brownian motion under $\mathbb{P}$ and L be the Radon-Nikodým density of $\mathbb{Q}$ w.r.t. $\mathbb{P}$. Then X is a $\mathbb{Q}$-Brownian motion if and only if X and L are $(\mathbb{P}, \mathbb{F})$-orthogonal martingales.

Example ([1] 1.7.8.2)

If $W = (X, Y)$ is a 2-dimensional Brownian motion starting from (a, b), the pair (x, l) where $x_t = W_t/|W_t|$ (stopped at the first time W vanishes) and $l_t = (Y_t, -X_t)$ satisfies the previous condition.

Bibliography

Bibliography

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