Predictable Representation Property and Girsanov's Theorem

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Predictable Representation Property

Predictable Representation Property

Let W be a real-valued Brownian motion and $\mathbb{F}^W$ its natural filtration.

Theorem (([1] 1.6.1.1)

Let M be square integrable $\mathbb{F}^W$-martingale. There exists a constant μ and a unique predictable process m in $L^2(W)$, such that

$$M_t = \mu + \int_0^t m_s \, dW_s, \quad \forall t \geq 0 \ .$$

Example

Consider the case $M_t = F(t, W_t)$, with F smooth (F is space-time harmonic, that is $\partial_t F + \frac{1}{2}\partial^2_{x^2} F = 0$).

Predictable Representation Property

Corollary

Every $\mathbb{F}^W$-local martingale admits a continuous version.

Corollary

Let W be a $\mathbb{G}$-Brownian motion with natural filtration $\mathbb{F}$. Then, for every square integrable $\mathbb{G}$-adapted process ψ,

$$\mathbb{E}\left[\int_0^t \psi_s \, dW_s \middle| \mathcal{F}_t\right] = \int_0^t \mathbb{E}\left[\psi_s \middle| \mathcal{F}_t\right] \, dW_s$$

where $\mathbb{E}\left[\psi_s \middle| \mathcal{F}_t\right]$ denotes the predictable version of the conditional expectation.

Example

- $M_t = \mathbb{E}[F|\mathcal{F}_t]$, with $F = \int_0^\infty h(s, W_s) \, ds$.
- $M_t = \mathbb{E}[h(W_T)|\mathcal{F}_t]$ for $t \leq T$ with $h \in C_b^1(\mathbb{R})$.

Change of Probability and Girsanov's Theorem

Change of Probability ([1] 1.7.1)

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with $\mathcal{F}_0$ trivial.

Proposition (([1] 1.7.1.1)

Let $\mathbb{P}$ and $\mathbb{Q}$ be two equivalent probabilities on $(\Omega, \mathcal{F}_T)$. Then, there exists a strictly positive $(\mathbb{P}, \mathbb{F})$-martingale $(L_t, t \leq T)$, such that

$$\mathbb{Q}|_{\mathcal{F}_t} = L_t \, \mathbb{P}|_{\mathcal{F}_t} \ ,$$

that is $\mathbb{E}_{\mathbb{Q}}[X] = \mathbb{E}_{\mathbb{P}}[L_t X]$ for any $\mathcal{F}_t$-measurable positive random variable X with $t \leq T$. Moreover, $L_0 = 1$ and $\mathbb{E}_P[L_t] = 1$, $\forall t \leq T$.

Definition

A probability $\mathbb{Q}$ on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is said to be *locally equivalent* to $\mathbb{P}$ if there exists a strictly positive $\mathbb{F}$-martingale L such that $\mathbb{Q}|_{\mathcal{F}_t} = L_t \, \mathbb{P}|_{\mathcal{F}_t}$, $\forall t \geq 0$. The martingale L is called the *Radon-Nikodým density* of $\mathbb{Q}$ w.r.t. $\mathbb{P}$.

Change of Probability ([1] 1.7.1)

Proposition (([1] 1.7.1.4)

Let $\mathbb{P}$ and $\mathbb{Q}$ be locally equivalent with Radon-Nikodým density L. Then, for any stopping time τ,

$$\mathbb{Q}|_{\mathcal{F}_\tau \cap \{\tau < \infty\}} = L_\tau \, \mathbb{P}|_{\mathcal{F}_\tau \cap \{\tau < \infty\}} \ .$$

Proposition (Bayes Formula ([1] 1.7.1.5)

Let $\mathbb{P}$ and $\mathbb{Q}$ be locally equivalent on $\mathcal{F}_T$ with Radon-Nikodým density L. Let X be a $\mathbb{Q}$-integrable $\mathcal{F}_T$-measurable random variable, then, for $t < T$

$$\mathbb{E}_\mathbb{Q}[X|\mathcal{F}_t] = \frac{1}{L_t}\mathbb{E}_\mathbb{P}[L_T X|\mathcal{F}_t] \ .$$

Change of Probability ([1] 1.7.1)

Proposition ([1] 1.7.1.6)

Let $\mathbb{P}$ and $\mathbb{Q}$ be locally equivalent with Radon-Nikodým density L. A process M is a $\mathbb{Q}$-martingale if and only if the process LM is a $\mathbb{P}$-martingale. By localization, this result remains true for local martingales.

Example

Find the Radon-Nikodým density L, for $\mathbb{Q} = h(W_T)\,\mathbb{P}$ with $h \in C_c^1(\mathbb{R}^+)$.

Exercise ([1] 1.7.1.7)

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space and denote by L the Radon-Nikodým density of $\mathbb{Q}$ with respect to $\mathbb{P}$.
Then, if $\tilde{\mathbb{F}}$ is a sub-filtration of $\mathbb{F}$, prove that $\mathbb{Q}|_{\tilde{\mathcal{F}}_t} = \tilde{L}_t\,\mathbb{P}|_{\tilde{\mathcal{F}}_t}$, where $\tilde{L}_t = \mathbb{E}_{\mathbb{P}}[L_t|\tilde{\mathcal{F}}_t]$.

Decomposition of $\mathbb{P}$-Martingales as $\mathbb{Q}$-semi-martingales

Theorem ([1] 1.7.2.1)

Let $\mathbb{P}$ and $\mathbb{Q}$ be locally equivalent with a continuous Radon-Nikodým density L. If M is a continuous $\mathbb{P}$-local martingale, then the process $\tilde{M}$ defined by

$$d\tilde{M} = dM - \frac{1}{L} d\langle M, L \rangle$$

is a continuous $\mathbb{Q}$-local martingale. If N is another continuous $\mathbb{P}$-local martingale $\langle M, N \rangle = \langle \tilde{M}, \tilde{N} \rangle = \langle M, \tilde{N} \rangle$.

Corollary ([1] 1.7.2.1)

We may write the process L as a Doléans-Dade martingale: $L_t = \mathcal{E}(\xi)$, where ξ is an $\mathbb{F}$-local martingale. The process $\tilde{M} = M - \langle M, \xi \rangle$ is a $\mathbb{Q}$-local martingale.

Bibliography

Bibliography

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