Introduction

Martingales and Stochastic Integration

Bernardo D’Auria
email: bernardo.dauria@uc3m.es
web: www.est.uc3m.es/bdauria
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Filtration and usual hypotheses
Fix a probability space \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\)

\(\mathbb{F}\) is called a *filtration* and it is a collection of \(\sigma\)-fields.

\[
\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}.
\]

\(\forall t \geq 0, \mathcal{F}_t \subset \mathcal{F}\), and for \(s \leq t\), \(\mathcal{F}_s \subset \mathcal{F}_t\).

Let \(\mathcal{F}_{t+} = \bigcap_{\epsilon > 0} \mathcal{F}_{t+\epsilon}\).

A filtration is *right-continuous* if \(\mathcal{F}_t = \mathcal{F}_{t+}\) for all \(t \geq 0\).

A filtration is *complete* if \(\mathcal{F}_t\) contains all null sets, for \(t \geq 0\).

A filtration that is right-continuous and complete is said to satisfy the *usual hypotheses*.

The *natural filtration* \(\mathbb{F}^X\) of a stochastic process \(X\) is the smallest filtration \(\mathbb{F}\) which satisfies the usual hypotheses and such that \(X\) is \(\mathbb{F}\)-adapted. In short \(\mathbb{F}^X = \sigma(X_s, s \leq t)\).
Martingales
Definition

An \( \mathbb{F} \)-adapted process \( M = (M_t, t \geq 0) \), is a \( \mathbb{F} \)-martingale if

\[
\mathbb{E}[|M_t|] < \infty \text{ for every } t \geq 0
\]
\[
\mathbb{E}[M_t|\mathcal{F}_s] = M_s \text{ a.s. for every } s < t.
\]

If the last condition is substituted for

\[
\mathbb{E}[M_t|\mathcal{F}_s] \leq M_s \quad \text{or} \quad \mathbb{E}[M_t|\mathcal{F}_s] \geq M_s \text{ a.s., for } s \leq t
\]

we speak of super-martingale or sub-martingale respectively.

Example

- Let \( W_t \) be a standard Brownian Motion, then \( W_t, W_t^2 - t \) and \( \exp\{aW_t - a^2 t/2\} \), for \( a \in \mathbb{R} \), are martingales.
- Let \( X_\infty \) an \( \mathcal{F}_\infty \)-measurable integrable r.v., then the process \( X \) defined as \( X_t := \mathbb{E}[X_\infty|\mathcal{F}_t] \) is a martingale.
Stopping times
Definition
An $\mathbb{R}^+ \cup \{+\infty\}$-valued random variable $\tau$ is a stopping time with respect to a given filtration $\mathbb{F}$ (in short, an $\mathbb{F}$-stopping time), if

$$\{\tau \leq t\} \in \mathcal{F}_t, \quad \forall t \geq 0.$$  

If $\tau$ is an $\mathbb{F}$-stopping time, the $\sigma$-algebra of events prior to $\tau$, $\mathcal{F}_\tau$ is defined as:

$$\mathcal{F}_\tau = \{A \in \mathcal{F}_\infty : A \cap \{\tau \leq t\} \in \mathcal{F}_t, \forall t \geq 0\}$$
Local Martingales
Definition
An adapted, right-continuous process \( M \) is a \( \mathbb{F} \)-local martingale if there exists a sequence of stopping times \((\tau_n)\) such that:

- The sequence \( \tau_n \) is increasing and \( \lim_n \tau_n = \infty \), a.s.
- For every \( n \), the stopped process \( M^{\tau_n} \) is an \( \mathbb{F} \)-martingale.

A sequence \((\tau_n)\) such that the two previous conditions hold is called a \textit{localizing or reducing sequence}. 
Proposition

A process $X$ is a sub-martingale (resp. a super-martingale) if and only if

$$X_t = M_t + A_t \quad (\text{resp. } X_t = M_t - A_t)$$

where

- $M$ is a local martingale,
- $A$ an increasing predictable process.
Square Integrable Martingales
Definition

We denote by $\mathbb{H}^2$ the space of $L^2$-bounded martingales, i.e. the space of $\mathcal{F}$-martingales $M$ such that

$$\sup_t \mathbb{E}[M_t^2] < \infty.$$ 

In addition we define:

$$\mathbb{H}^2 := \{ M \in \mathbb{H}^2 : M \text{ is continuous} \}$$

$$\mathbb{H}_0^2 := \{ M \in \mathbb{H}^2 : M_0 = 0 \}$$
A square integrable martingale $M$ is also *uniformly integrable* (u.i.) (i.e. $\mathbb{E}[|M_t|; |M_t| > a] \xrightarrow{a \to \infty} 0$ uniformly on $t$) and therefore

$$M_t \xrightarrow{L^1} M_\infty \quad \text{and} \quad M_t = \mathbb{E}[M_\infty | \mathcal{F}_t].$$

In addition

$$\mathbb{E}[M_\infty^2] = \lim_{t \to \infty} \mathbb{E}[M_t^2] = \sup_t \mathbb{E}[M_t^2] < \infty$$

From $M_t^2 \leq \mathbb{E}[M_\infty^2 | \mathcal{F}_t]$, it follows that $M^2 = (M_t^2, t \geq 0)$ is u.i.

**Example**

- The Brownian motion does not belong to $H^2$, however, it belongs to $H^2[0, T]$ for all $T > 0$.
- The martingale $M_t = \exp\{\lambda W_t - \lambda^2 t/2\}$ is not u.i. on $[0, \infty[$.
Accordingly to the previous definition we define the spaces

$$H^p = \{ M \text{ is an } \mathcal{F}\text{-adapted martingale : } \| \sup_t |M_t| \|_p < \infty \} \quad p \geq 1,$$

where $$\|X\|_p = (\mathbb{E}[|X|^p])^{1/p}$$.

By Doob’s inequality, for $$p > 1$$,

$$\| \sup_{t \leq T} |M_t| \|_p \leq \frac{p}{p - 1} \sup_{t \leq T} \|M_t\|_p,$$

it follows that

$$\sup_t |M_t| \in L^p \iff M_\infty \in L^p$$

and one can identify the space $$H^p$$ with the space $$L^p(\mathcal{F}_\infty)$$, for $$p > 1$$.

**Note:** For $$p = 1$$ this is not true. There are $$L^1$$ bounded elements that are not in $$H^1$$. 
Continuous Semi-martingales
Definition

A $d$-dimension continuous semi-martingale is an $\mathbb{R}^d$-valued process $X$ such that each component $X^i$ admits a decomposition

$$X^i = M^i + A^i$$

where

- $M^i$ is a continuous local martingale vanishing at zero
- $A^i$ is a continuous adapted process with locally finite variation

The continuity property assures that the decomposition is unique.
Brackets of Continuous Local Martingales ([1] 1.3.1)

Definition
If $M$ is a continuous local martingale, there exists a unique continuous increasing process $\langle M \rangle$, called the bracket (or predictable quadratic variation) of $M$ such that

$$M^2 - \langle M \rangle = (M_t^2 - \langle M \rangle_t, t \geq 0)$$

is a continuous local martingale.

Definition
If $M, N$ are two continuous local martingales, there exists a unique continuous increasing process $\langle M, N \rangle$, called the predictable bracket (or the predictable covariation process) of $M$ and $N$, such that

$$MN - \langle M, N \rangle = (M_t N_t - \langle M, N \rangle_t, t \geq 0)$$

is a continuous local martingale.
It can be shown that

\[
\langle M \rangle_t = \lim_{||\Pi|| \to 0} \frac{|\Pi|}{||\Pi||} \sum_{i=1}^{||\Pi||} (M_{t_i} - M_{t_{i-1}})^2,
\]

where for an interval \([0, t]\), we define a partition as the set \(\Pi = \{0 = t_0 < t_1 < t_2 < \ldots < t_n = t\}\) for some \(n\) and \(|\Pi| = n\), 
\(||\Pi|| = \max_{1 \leq i \leq n} (t_i - t_{i-1})\).

**Example**

If \(W\) is a Brownian motion

\[
\langle W \rangle_t = \lim_{||\Pi|| \to 0} \frac{|\Pi|}{||\Pi||} \sum_{i=1}^{||\Pi||} (W_{t_i} - W_{t_{i-1}})^2 = t.
\]

The limit here is in \(L^2\) sense, (that implies the limit in probability), and it can be shown that the limit is almost surely if \(\sum_n |\Pi_n| < \infty\).
Predictable Brackets

We have that

\[ \langle M \rangle = \langle M, M \rangle \]

and

\[
\langle M, N \rangle = \frac{1}{2} \left( \langle M + N \rangle - \langle M \rangle - \langle N \rangle \right) = \frac{1}{4} \left( \langle M + N \rangle - \langle M - N \rangle \right)
\]

When the bracket \( \langle M, N \rangle \) is equal to 0, the product \( MN \) is a local martingale and \( M \) and \( N \) are said to be **orthogonal**.
The space $L^2(M)$

**Definition**

Let $M \in \mathcal{H}^2$, then the space $L^2(M, \mathbb{F})$, or simply $L^2(M)$, is defined as

$$L^2(M) = \{ K \text{ is a } \mathbb{F} \text{-progressively measurable : } \mathbb{E}\left[ \int_0^\infty K_t^2 d\langle M \rangle_t \right] < \infty \}.$$

A real-valued process $K$ is *progressively measurable* with respect to a given filtration $\mathbb{F} = (\mathcal{F}_t, t \geq 0)$. if, for every $t \geq 0$, the map $(\omega, s) \to K_s(\omega)$ form $\Omega \times [0, t]$ into $\mathbb{R}$ is $\mathcal{F}_t \times \mathcal{B}([0, t])$-measurable.

Any cád (or cág) $\mathbb{F}$-adapted process is progressively measurable.
Definition

The bracket (or the predictable quadratic covariation) \( \langle X, Y \rangle \) of two continuous semi-martingales \( X \) and \( Y \) is defined as the bracket of the local martingale parts \( M^X \) and \( M^Y \).
Stochastic Integral
Definition

Let $K$ be $\mathbb{F}$-predictable process and $M \in \mathbb{H}^2$ then the Itô stochastic integral $K \star M = (K \star M_t, t \geq 0)$, is defined as

$$K \star M_t = \int_0^t K_s \, dM_s := \operatorname{p-lim} \sum_{i=1}^{||\Pi||} K_{t_{i-1}} (M_{t_i} - M_{t_{i-1}}).$$

In addition $K \star M$ is an $\mathbb{F}$-martingale vanishing at 0.

For an interval $[0, t]$, we define a partition as the set

$\Pi = \{0 = t_0 < t_1 < t_2 < \ldots < t_n = t\}$ for some $n$ and $||\Pi|| = n,$

$||\Pi|| = \max_{1 \leq i \leq n} (t_i - t_{i-1}).$
Properties of the Stochastic Integral

Linearity:
\[
\int_0^t (aH_s + bK_s) \, dM_s = a \int_0^t H_s \, dM_s + b \int_0^t K_s \, dM_s.
\]

Itô’s Isometry:
\[
\mathbb{E} \left[ \left( \int_0^t K_s \, dM_s \right)^2 \right] = \mathbb{E} \left[ \int_0^t K_s^2 \, d\langle M \rangle_s \right].
\]
Formulas for the Stochastic Integral

Integration by Parts:

\[ X_t Y_t = X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + \langle X, Y \rangle \]

Itô’s Formula: with \( F \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^d, \mathbb{R}) \),

\[ F(t, X_t) = F(0, X_0) + \int_0^t \partial_t F(s, X_s) ds + \sum_{i=1}^d \int_0^t \partial_{x_i} F(s, X_s) dX_s^i \]

\[ + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \partial_{x_i x_j} F(s, X_s) d\langle X^i, X^j \rangle_s \]
Doléans-Dade Exponential
Doléans-Dade Exponential ([1] 1.5.7)

Definition
Let $M$ be a continuous local martingale and $\lambda \in \mathbb{R}$, then the Doléans-Dade Exponential of $\lambda M$ is a positive local martingale, $\mathcal{E}(\lambda M) = (\mathcal{E}(\lambda M)_t, t \geq 0)$ defined as

$$\mathcal{E}(\lambda M)_t := \exp\{\lambda M_t - \frac{\lambda^2}{2} \langle M \rangle_t\}.$$ 

Proposition
If $\lambda \in L^2(M)$, the process $\mathcal{E}(\lambda M)$ is the unique solution of the stochastic differential equation

$$dY_t = Y_t d(\lambda M)_t, \quad Y_0 = 1.$$
Doléans-Dade Exponential for semi-martingales

**Definition**

Let \( X \) be a continuous semi-martingale, then the *Doléans-Dade Exponential* of \( X \) is defined as the unique solution of

\[
Z_t = 1 + \int_0^t Z_s \, dX_s ,
\]

and it is given by

\[
\mathcal{E}(X)_t := \exp\{X_t - \frac{1}{2}\langle X \rangle_t\} .
\]

**Note:** \( \mathcal{E}(\lambda M)_t \mathcal{E}(\mu M)_t \neq \mathcal{E}((\lambda + \mu)M)_t \). In general

\[
\mathcal{E}(X)_t \mathcal{E}(Y)_t = \mathcal{E}(X + Y + \langle X, Y \rangle)_t
\]

leads to \( \mathcal{E}(\lambda M)_t \mathcal{E}(\mu M)_t = \mathcal{E}((\lambda + \mu)M + \lambda \mu \langle M \rangle)_t \).