Introduction

Martingales and Stochastic Integration

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July 3, 2017

ICMAT / UC3M

Fitration and usual hypotheses

Fitration and usual hypotheses ([1] 1.1.10)

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$

$\mathbb{F}$ is called a *filtration* and it is a collection of σ-fields.

$$\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0} \ .$$

$\forall t \geq 0$, $\mathcal{F}_t \subset \mathcal{F}$, and for $s \leq t$, $\mathcal{F}_s \subset \mathcal{F}_t$.

Let $\mathcal{F}_{t+} = \bigcap_{\epsilon > 0} \mathcal{F}_{t+\epsilon}$.

A filtration is *right-continuous* if $\mathcal{F}_t = \mathcal{F}_{t+}$ for all $t \geq 0$.
A filtration is *complete* if $\mathcal{F}_t$ contains all null sets, for $t \geq 0$.
A filtration that is right-continuous and complete is said to satisfy the *usual hypotheses*.

The *natural filtration* $\mathbb{F}^X$ of a stochastic process X is the smallest filtration $\mathbb{F}$ which satisfies the usual hypotheses and such that X is $\mathbb{F}$-adapted. In short $\mathbb{F}^X = \sigma(X_s, s \leq t)$.

Martingales

Martingales ([1] 1.2.1)

Definition

An $\mathbb{F}$-adapted process $M = (M_t, t \geq 0)$, is a $\mathbb{F}$-martingale if

$$\mathbb{E}[|M_t|] < \infty \text{ for every } t \geq 0$$

$$\mathbb{E}[M_t|\mathcal{F}_s] = M_s \text{ a.s. for every } s < t.$$

If the last condition is substituted for

$$\mathbb{E}[M_t|\mathcal{F}_s] \leq M_s \quad \text{or} \quad \mathbb{E}[M_t|\mathcal{F}_s] \geq M_s \quad \text{a.s., for } s \leq t$$

we speak of *super-martingale* or *sub-martingale* respectively.

Example

- Let W_t be a standard Brownian Motion, then W_t, $W_t^2 - t$ and $\exp\{aW_t - a^2 t/2\}$, for $a \in \mathbb{R}$, are martingales.
- Let X_∞ an $\mathcal{F}_\infty$-measurable integrable r.v., then the process X defined as $X_t := \mathbb{E}[X_\infty|\mathcal{F}_t]$ is a martingale.

Stopping times

Definition

An $\mathbb{R}^+ \cup \{+\infty\}$-valued random variable τ is a *stopping time* with respect to a given filtration $\mathbb{F}$ (in short, an $\mathbb{F}$-stopping time), if

$$\{\tau \leq t\} \in \mathcal{F}_t, \quad \forall t \geq 0 \ .$$

If τ is an $\mathbb{F}$-stopping time, the σ-algebra of events prior to τ, $\mathcal{F}_\tau$ is defined as:

$$\mathcal{F}_\tau = \{A \in \mathcal{F}_\infty : A \cap \{\tau \leq t\} \in \mathcal{F}_t, \forall t \geq 0\}$$

Local Martingales

Local Martingales ([1] 1.2.4)

Definition

An adapted, right-continuous process M is a $\mathbb{F}$-*local martingale* if there exists a sequence of stopping times (τ_n) such that:

The sequence τ_n is increasing and $\lim_n \tau_n = \infty$, a.s.

For every n, the stopped process M^{τ_n} is an $\mathbb{F}$-martingale.

A sequence (τ_n) such that the two previous conditions hold is called a *localizing or reducing sequence*.

Doob-Mayer decomposition

Proposition

A process X is a sub-martingale (resp. a super-martingale) if and only if

$$X_t = M_t + A_t \quad (resp.\ X_t = M_t - A_t)$$

where

M is a local martingale,

A an increasing predictable process.

Square Integrable Martingales

Square Integrable Martingales ([1] 1.2.2)

Definition

We denote by $\mathbb{H}^2$ the space of L^2-bounded martingales, i.e. the space of $\mathbb{F}$-martingales M such that

$$\sup_t \mathbb{E}[M_t^2] < \infty \ .$$

In addition we define:

$$\mathbf{H}^2 := \{M \in \mathbb{H}^2 : M \text{ is continuous}\}$$
$$\mathbf{H}_0^2 := \{M \in \mathbf{H}^2 : M_0 = 0\}$$

Square Integrable Martingales

A square integrable martingale M is also *uniformly integrable* (u.i.) (i.e. $\mathbb{E}[|M_t|; |M_t| > a] \stackrel{a\to\infty}{\longrightarrow} 0$ uniformly on t) and therefore

$$M_t \longrightarrow_{L^1} M_\infty \quad \text{and} \quad M_t = \mathbb{E}[M_\infty | \mathcal{F}_t] \ .$$

In addition

$$\mathbb{E}[M_\infty^2] = \lim_{t\to\infty} \mathbb{E}[M_t^2] = \sup_t \mathbb{E}[M_t^2] < \infty$$

From $M_t^2 \leq \mathbb{E}[M_\infty^2 | \mathcal{F}_t]$, it follows that $M^2 = (M_t^2, t \geq 0)$ is u.i.

Example

- The Brownian motion does not belong to $\mathbf{H}^2$, however, it belongs to $\mathbf{H}^2[0, T]$ for all $T > 0$.
- The martingale $M_t = \exp\{\lambda W_t - \lambda^2 t/2\}$ is not u.i. on $[0, \infty[$.

Spaces of Martingales

Accordingly to the previous definition we define the spaces

$$\mathbb{H}^p = \{M \text{ is an } \mathbb{F}\text{-adapted martingale} : ||\sup_t |M_t|\,||_p < \infty\} \quad p \geq 1\,,$$

where $||X||_p = (\mathbb{E}[|X|^p])^{1/p}$.

By Doob's inequality, for $p > 1$,

$$||\sup_{t \leq T} |M_t|\,||_p \leq \frac{p}{p-1} \sup_{t \leq T} ||M_t||_p \;,$$

it follows that

$$\sup_t |M_t| \in L^p \iff M_\infty \in L^p$$

and one can identify the space $\mathbb{H}^p$ with the space $L^p(\mathcal{F}_\infty)$, for $p > 1$.

Note: For $p = 1$ this is not true. There are L^1 bounded elements that are not in $\mathbb{H}^1$.

Continuous Semi-martingales

Continuous Semi-martingales ([1] 1.3)

Definition

A d-dimension *continuous semi-martingale* is an $\mathbb{R}^d$-valued process X such that each component X^i admits a decomposition

$$X^i = M^i + A^i$$

where

M^i is a continuous local martingale vanishing at zero
A^i is a continuous adapted process with locally finite variation

The continuity property assures that *the decomposition is unique*.

Brackets of Continuous Local Martingales ([1] 1.3.1)

Definition

If M is a continuous local martingale, there exists a unique continuous increasing process $\langle M \rangle$, called *the bracket* (or predictable quadratic variation) of M such that

$$M^2 - \langle M \rangle = (M_t^2 - \langle M \rangle_t,\ t \geq 0)$$

is a continuous local martingale.

Definition

If M, N are two continuous local martingales, there exists a unique continuous increasing process $\langle M, N \rangle$, called *the predictable bracket* (or the predictable covariation process) of M and N, such that

$$M\,N - \langle M, N \rangle = (M_t N_t - \langle M, N \rangle_t,\ t \geq 0)$$

is a continuous local martingale.

Brackets of Continuous Local Martingales ([1] 1.3.1)

It can be shown that

$$\langle M \rangle_t = \operatorname*{p\text{-}lim}_{||\Pi|| \to 0} \sum_{i=1}^{|\Pi|} (M_{t_i} - M_{t_{i-1}})^2 \ ,$$

where for an interval $[0, t]$, we define a partition as the set $\Pi = \{0 = t_0 < t_1 < t_2 < \ldots < t_n = t\}$ for some n and $|\Pi| = n$, $||\Pi|| = \max_{1 \le i \le n} (t_i - t_{i-1})$.

Example

If W is a Brownian motion

$$\langle W \rangle_t = \lim_{||\Pi|| \to 0} \sum_{i=1}^{|\Pi|} (W_{t_i} - W_{t_{i-1}})^2 = t \ .$$

The limit here is in L^2 sense, (that implies the limit in probability), and it can be shown that the limit is almost surely if $\sum_n |\Pi_n| < \infty$.

Predictable Brackets

We have that

$$\langle M \rangle = \langle M, M \rangle$$

and

$$\langle M, N \rangle = \frac{1}{2} \left(\langle M + N \rangle - \langle M \rangle - \langle N \rangle \right) = \frac{1}{4} \left(\langle M + N \rangle - \langle M - N \rangle \right)$$

When the bracket $\langle M, N \rangle$ is equal to 0, the product $M\,N$ is a local martingale and M and N are said to be *orthogonal*.

The space $L^2(M)$

Definition

Let $M \in \mathbf{H}^2$, then the space $L^2(M, \mathbb{F})$, or simply $L^2(M)$, is defined as

$$L^2(M) = \{K \text{ is a } \mathbb{F}\text{-progressively measurable} : \mathbb{E}[\int_0^\infty K_t^2 d\langle M\rangle_t] < \infty\} .$$

A real-valued process K is *progressively measurable* with respect to a given filtration $\mathbb{F} = (\mathcal{F}_t, t \geq 0)$. if, for every $t \geq 0$, the map $(\omega, s) \to K_s(\omega)$ form $\Omega \times [0, t]$ into $\mathbb{R}$ is $\mathcal{F}_t \times \mathcal{B}([0, t])$-measurable.

Any cád (or cág) $\mathbb{F}$-adapted process is progressively measurable.

Brackets of Continuous Semi-martingales ([1] 1.3.2)

Definition

The bracket (or the predictable quadratic covariation) $\langle X, Y \rangle$ of two continuous semi-martingales X and Y is defined as the bracket of the local martingale parts M^X and M^Y.

Stochastic Integral

Stochastic Integral ([2] IV Prop. (2.13))

Definition

Let K be $\mathbb{F}$-predictable process and $M \in \mathbf{H}^2$ then the Itô *stochastic integral* $K \star M = (K \star M_t, t \geq 0)$, is defined as

$$K \star M_t = \int_0^t K_s \, dM_s := \operatorname*{p-lim}_{||\Pi|| \to 0} \sum_{i=1}^{|\Pi|} K_{t_{i-1}} (M_{t_i} - M_{t_{i-1}}) \; .$$

In addition $K \star M$ is an $\mathbb{F}$-martingale vanishing at 0.

For an interval $[0, t]$, we define a partition as the set $\Pi = \{0 = t_0 < t_1 < t_2 < \ldots < t_n = t\}$ for some n and $|\Pi| = n$, $||\Pi|| = \max_{1 \leq i \leq n} (t_i - t_{i-1})$.

Properties of the Stochastic Integral

Linearity:

$$\int_0^t (aH_s + bK_s)\, dM_s = a \int_0^t H_s\, dM_s + b \int_0^t K_s\, dM_s \ .$$

Itô's Isometry:

$$\mathbb{E}\Big[\Big(\int_0^t K_s\, dM_s \Big)^2 \Big] = \mathbb{E}\Big[\int_0^t K_s^2 d\langle M \rangle_s \Big]$$

Formulas for the Stochastic Integral

Integration by Parts:

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + \langle X, Y \rangle$$

Itô's Formula: with $F \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^d, \mathbb{R})$,

$$\begin{aligned} F(t, X_t) =& F(0, X_0) + \int_0^t \partial_t F(s, X_s) ds + \sum_{i=1}^{d} \int_0^t \partial_{x_i} F(s, X_s) dX_s^i \\ &+ \frac{1}{2} \sum_{i,j=1}^{d} \int_0^t \partial^2_{x_i, x_j} F(s, X_s) d\langle X^i, X^j \rangle_s \end{aligned}$$

Doléans-Dade Exponential

Doléans-Dade Exponential ([1] 1.5.7)

Definition

Let M be a continuous local martingale and $\lambda \in \mathbb{R}$, then the *Doléans-Dade Exponential* of λM is a positive local martingale, $\mathcal{E}(\lambda M) = (\mathcal{E}(\lambda M)_t, t \geq 0)$ defined as

$$\mathcal{E}(\lambda M)_t := \exp\{\lambda M_t - \frac{\lambda^2}{2}\langle M \rangle_t\} \ .$$

Proposition

If $\lambda \in L^2(M)$, the process $\mathcal{E}(\lambda M)$ is the unique solution of the stochastic differential equation

$$dY_t = Y_t d(\lambda M)_t, \quad Y_0 = 1 \ .$$

Doléans-Dade Exponential for semi-martingales

Definition

Let X be a continuous semi-martingale, then the *Doléans-Dade Exponential* of X is defined as the unique solution of

$$Z_t = 1 + \int_0^t Z_s \, dX_s \ ,$$

and it is given by

$$\mathcal{E}(X)_t := \exp\{X_t - \frac{1}{2}\langle X \rangle_t\} \ .$$

Note: $\mathcal{E}(\lambda M)_t \, \mathcal{E}(\mu M)_t \neq \mathcal{E}((\lambda + \mu)M)_t$. In general

$$\mathcal{E}(X)_t \, \mathcal{E}(Y)_t = \mathcal{E}(X + Y + \langle X, Y \rangle)_t$$

leads to $\mathcal{E}(\lambda M)_t \, \mathcal{E}(\mu M)_t = \mathcal{E}((\lambda + \mu)M + \lambda\mu\langle M \rangle)_t$.

Bibliography

Bibliography

- M. Jeanblanc, M. Yor and M. Chesney (2009). *Mathematical methods for financial markets*. Springer, London.
- D. Revuz and M. Yor (1999). *Continuous Martingales and Brownian Motion.* Springer Verlag, Berlin. 3rd ed.