Notes

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1 Application of the Stochastic Calculus

1.1 The Geometric Brownian Motion

In this section we look for the solution of the following SDE

$$\frac{dX(t)}{X(t)} = \mu\, dt + \sigma\, dB(t) \tag{1}$$

that can also be rewritten as

$$dX(t) = \mu\, X(t)\, dt + \sigma\, X(t)\, dB(t). \tag{2}$$

We use as test function $X(t) = f(t, B(t))$ and applying the Itôformula, we get

$$df(t, B(t)) = [f_t(t, B(t)) + \frac{1}{2} f_{xx}(t, B(t))]\, dt + f_x(t, B(t))\, dB(t). \tag{3}$$

Matching the coefficients of dt and $dB(t)$ we obtain

$$\mu f(x, t) = f_t(t, x) + \frac{1}{2} f_{xx}(t, x) \tag{4}$$

$$\sigma f(x, t) = f_x(t, x) \tag{5}$$

$$\tag{6}$$

and taking the derivative of (5) and substituting it in (4) we get

$$f_t(t, x) = \left[\mu - \frac{\sigma^2}{2}\right] f(t, x)$$

that solved gives

$$f(t, x) = f(0, x)\, e^{\left(\mu - \frac{\sigma^2}{2}\right) t}.$$

Substituting last expression in (5) we get

$$f_x(0, x) = \sigma f(0, x)$$

that has solution

$$f(0, x) = f(0, 0) e^{\sigma\, x}.$$

Finally the solution of (2) is given by

$$X(t) = f(t, B(t)) = X(0)\, e^{\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma\, B(t)}.$$

Alternative derivation.

Another way to get a solution of (5) is by computing the correct differential of the function $\ln(X(t)$ that differs from $dX(t)/X(t)$ of the classical calculus.

Using Itôformula, we have that

$$d \ln(X(t)) = \frac{dX(t)}{X(t)} - \frac{(dX(t))^2}{2X^2(t)},$$

and substituting the expressions of $dX(t)$ and $(dX(t))^2$ obtained from (2)

$$\begin{aligned} dX(t) &= \mu\, X(t)\, dt + \sigma\, X(t)\, dB(t) \\ (dX(t))^2 &= \mu^2\, X^2(t)\, (dt)^2 + \sigma^2\, X^2(t)\, (dB(t))^2 + 2\mu\sigma X^2(t)\, dB(t) \times dt \\ &= \sigma^2\, X^2(t)\, dt \end{aligned}$$

we obtain

$$d\ln(X(t)) = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma\, dB(t)$$

that integrated yields again to

$$X(t) = X(0)\, e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\, B(t)}.$$

1.2 The Uhlenbeck-Ornstein process

In this section we look for the solution, $X(t)$, of the following SDE

$$dX(t) = -\alpha\, X(t)\, dt + \sigma\, dB(t),$$

with α and σ two positive constants.

Use the test function $X(t) = f(t) = a(t)\left[c + \int_0^t b(s)\, dB(s)\right]$ whose differential is equal to

$$dX(t) = \frac{a'(t)}{a(t)} X(t)\, dt + a(t)\, b(t)\, dB(t),$$

where we used the formula for the differential of the product of functions.

Matching the coefficients of dt and $dB(t)$, we get

$$\frac{a'(t)}{a(t)} = -\alpha \quad \text{and} \quad a(t)\, b(t) = \sigma$$

that solved give

$$a(t) = a(0)\, e^{-\alpha\, t} \quad \text{and} \quad b(t) = \frac{\sigma}{a(t)} = \frac{\sigma}{a(0)} e^{\alpha\, t}$$

and setting $f(0) = X(0)$ we finally obtain

$$X(t) = X(0)\, e^{-\alpha\, t} + \sigma \int_0^t e^{-\alpha(t-s)}\, dB(s).$$

By the Itôisometry, we obtain that if $X(0)$ is independent of $B(t)$ and normally distributed (including the deterministic degenerate normal distribution), then $X(t)$ is a Gaussian process with mean and covariance functions

$$\begin{aligned}
\mathbb{E}[X(t)] &= \mathbb{E}[X(0)]\, e^{-\alpha\, t} \to 0 \qquad \text{as } t \to \infty \\
\mathbb{C}\text{ov}[X(t), X(s)] &= \mathbb{V}\text{ar}[X(0)]\, e^{-\alpha\,(t+s)} + \sigma^2 \int_0^{t\wedge s} e^{-\alpha(t+s-2u)}\, du \\
\mathbb{V}\text{ar}[X(t)] &= \mathbb{V}\text{ar}[X(0)]\, e^{-2\alpha\, t} + \sigma^2 \int_0^t e^{-2\alpha(t-u)}\, du \to \frac{\sigma^2}{2\alpha} \qquad \text{as } t \to \infty
\end{aligned}$$

Therefore we see that the O-U process admits a stationary distribution and we can construct a stationary version of the process by setting $X(0) \sim \mathrm{N}(0, \frac{\sigma^2}{2\alpha})$.

1.3 The Brownian Bridge

In this section we look for the solution, $X(t)$, of the following SDE

$$dX(t) = -\frac{X(t)}{1-t}\, dt + dB(t),$$

with $X(0) = 0$.

Use the test function $X(t) = f(t) = a(t)\left[c + \int_0^t b(s)\, dB(s)\right]$ whose differential is equal to

$$dX(t) = \frac{a'(t)}{a(t)} X(t)\, dt + a(t)\, b(t)\, dB(t).$$

Matching the coefficients of dt and $dB(t)$, we get

$$\frac{a'(t)}{a(t)} = -\frac{1}{1-t} \quad \text{and} \quad a(t)\,b(t) = 1$$

that solved give

$$a(t) = 1-t \quad \text{and} \quad b(t) = \frac{1}{1-t}$$

and setting $X(0) = 0$ we finally obtain

$$X(t) = \int_0^t \frac{1-t}{1-s}\,dB(s). \tag{7}$$

By the Itôisometry, we obtain that $X(t)$ is a Gaussian process with mean and covariance functions

$$\mathbb{E}[X(t)] = 0 \quad \text{and} \quad \mathbb{C}\mathrm{ov}[X(t), X(s)] = (1 - t \vee s)(t \wedge s)$$

that coincide with the ones of the Brownian Bridge. Since two Gaussian processes with identical mean and covariance functions are equal in distribution we see that equation (7) gives an alternative representation of the Brownian Bridge process.