Notes

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1 Stochastic Calculus

In 1900, Bachelier proposed for the Paris stock exchange a model for the fluctuations affecting the price $X(t)$ of an asset that was given by the Brownian motion. By calling $dX(t)$ the infinitesimal variation of the price, he proposed

$$dX(t) = \mu dt + \sigma\, dB(t). \tag{1}$$

Even if at the moment equation (1) does not have a precise meaning, one can guess that it has as solution the (μ, σ)-BM, i.e.

$$X(t) = \mu\, t + \sigma\, B(t).$$

The problem of last solution is that

$$\mathbb{P}\{X(t) < 0\} > 0$$

especially when $\mu < 0$, where this probability increases to 1 as $t \to \infty$.

A solution to this modeling problem could be find by changing the absolute differential changing in the price $dX(t)$ by the relative changing, i.e. $\frac{dX(t)}{X(t)}$, in such a way to obtain the following Stochastic Differential Equation (SDE),

$$dX(t) = \mu\, X(t)dt + \sigma\, X(t)\, dB(t). \tag{2}$$

Again, at the moment we have no precise meaning for the SDE (2), but we could naively try to guess that the following steps will give the right solution to it.

$$\frac{dX(t)}{X(t)} = \mu\, dt + \sigma\, dB(t) \quad \Rightarrow \quad d\ln\{X(t)\} = \mu\, dt + \sigma\, dB(t) \quad \Rightarrow \quad X(t) = X(0)\, e^{\mu\, t + \sigma\, B(t)}. \quad \textbf{WRONG!!}$$

We will see after a precise definition of what we mean by a SDE that last guess is indeed a wrong guess and that more care we need when handling with stochastic calculus.

1.1 SDE seen as Stochastic Integral Equations

Before starting the definition of the stochastic integral we start noticing that it has no meaning to speak about the differential of the Brownian Motion, i.e. $dB(t)$. Indeed the Brownian motion has the property to have almost surely all the sample functions continuous but nowhere differentiable. To have just a felling of this let us look the following limit

$$\lim_{h\to 0} \frac{B(h) - B(0)}{\sqrt{h}} \overset{\mathcal{L}^2}{=} X \sim N(0, 1)$$

and from this we get that there is no defined limit for $(B(h) - B(0))/h$. Indeed for any $a > 0$ we have

$$\lim_{h\to 0} \mathbb{P}\{\frac{B(h) - B(0)}{h} > a\} = \lim_{h\to 0} \mathbb{P}\{\frac{B(h) - B(0)}{\sqrt{h}} > a\sqrt{h}\} = \frac{1}{2}$$

therefore if there were a limit r.v. it would take only values $\pm\infty$, each with probability 1/2.

Since we cannot speak about the differential Brownian Motion we need an alternative interpretation for the following SDE

$$dX(t) = a(t, X(t))dt + b(t, X(t))\, dB(t) \tag{3}$$

with $a(t, x)$ and $b(t, x)$ two given random functions.

The interpretation is the following, the stochastic process $X(t)$ is a solution of the SDE (3) if it satisfies for each $t \geq$ the following integral relation

$$X(t) = X(0) + \int_0^t a(s, X(s))ds + \int_0^t b(s, X(s))\, dB(s) \tag{4}$$

and we are left only with the definition of the meaning of the stochastic integral $\int_0^t b(s, X(s))\, dB(s)$.

1.2 Not uniqueness of the Riemann sum

A first attempt in the definition of the stochastic integral could be to define it as a limit of a Riemann sum. To this aim, given an interval $[0,t]$ let define $\Pi_n = \{0 = t_0 < t_1 < t_2 < \ldots < t_n = t\}$ be a partition of it in n sub intervals, and let define

$$|\Pi_n| = n, \quad ||\Pi_n|| = \max_{1\leq k\leq n} (t_k - t_{k-1})\,.$$

In addition we denote by $\mathcal{P}_n$ the set of all partition of order n of the interval $[0,t]$ and by $\mathcal{P} = \bigcup_{n>0} \mathcal{P}_n$, the set of all partitions.

In general, if we have two continuous functions $f(t)$ and $g(t)$, mutually integrable (for this it is enough to have $g(t)$ of BV, see later for more details), the Riemann-Stieltjes integral is defined as

$$\int_0^t f(s)dg(s) = \lim_{||\Pi||\to 0} \sum_{k=1}^{|\Pi|} f(\xi_k)[g(t_k) - g(t_{k-1})],$$

where ξ_k is any point in the interval $[t_{k-1},\, t_k]$.

In the following we check that, as for the definition of the stochastic integral, it is crucial how to choose the sequence of points ξ_k, because to different choices correspond different values of the integral.

In the following we are going to compute $\int_0^t B(s)\,dB(s)$, with $B(t)$ the standard Brownian Motion, using two different extreme choices for ξ_k, i.e. $\xi_k = t_{k-1}$ and $\xi_k = t_k$.

Case 1: $\xi_k = t_{k-1}$

$$\int_0^t B(s)\,dB(s) = \lim_{||\Pi||\to 0} \sum_{k=1}^{|\Pi|} B(t_{k-1})[B(t_k) - B(t_{k-1})].$$

Having that

$$B(t_k)B(t_{k-1}) = \frac{B^2(t_k) + B^2(t_{k-1})}{2} - \frac{[B(t_k) - B(t_{k-1})]^2}{2} \tag{5}$$

we obtain

$$\begin{aligned}\int_0^t B(s)\,dB(s) &= \lim_{||\Pi||\to 0} \left\{ \frac{1}{2}\sum_{k=1}^{|\Pi|}[B^2(t_k) - B^2(t_{k-1})] - \frac{1}{2}\sum_{k=1}^{|\Pi|}[B(t_k) - B(t_{k-1})]^2 \right\} \\ &= \frac{B^2(t)}{2} - \lim_{||\Pi||\to 0} \frac{1}{2}\sum_{k=1}^{|\Pi|}[B(t_k) - B(t_{k-1})]^2 \overset{\mathcal{L}^2}{\to} \frac{B^2(t)}{2} - \frac{t}{2},\end{aligned}$$

where we used the fact that

$$\sum_{k=1}^{|\Pi|}[B(t_k) - B(t_{k-1})]^2 \xrightarrow[||\Pi||\downarrow 0]{\mathcal{L}^2} t, \tag{6}$$

to be proved later. Actually for the Brownian Motion, the limit in (6) is valid also with probability 1.

Case 2: $\xi_k = t_k$

$$\int_0^t B(s)\,dB(s) = \lim_{||\Pi||\to 0} \sum_{k=1}^{|\Pi|} B(t_k)[B(t_k) - B(t_{k-1})].$$

Again using (5) we obtain

$$\begin{aligned}\int_0^t B(s)\,dB(s) &= \lim_{||\Pi||\to 0} \left\{ \frac{1}{2}\sum_{k=1}^{|\Pi|}[B^2(t_k) - B^2(t_{k-1})] + \frac{1}{2}\sum_{k=1}^{|\Pi|}[B(t_k) - B(t_{k-1})]^2 \right\} \\ &= \frac{B^2(t)}{2} + \lim_{||\Pi||\to 0} \frac{1}{2}\sum_{k=1}^{|\Pi|}[B(t_k) - B(t_{k-1})]^2 \overset{\mathcal{L}^2}{\to} \frac{B^2(t)}{2} + \frac{t}{2},\end{aligned}$$

where we used again the convergence relation (6).

1.3 Total Variation and Quadratic Variation

In this section we prove that the $\mathcal{L}^2$ convergence in (6) holds and that its limit is given by the quadratic variation of the Brownian motion over the interval $[0,t]$

Definition 1. *Given a function $f(t)$, $t \geq 0$, the total variation of f over the interval $[0,t]$, $V_t(f)$, is defined as*

$$V_t(f) = \sup_{\Pi \in \mathcal{P}} \sum_{k=1}^{|\Pi|} |f(t_k) - f(t_{k-1})|. \tag{7}$$

If $V_t(f) < \infty$ for any $t \geq 0$ we say that f is of Bounded Variation *and we denote it by writing $f \in BV$.*

Definition 2. *Given a function $f(t)$, $t \geq 0$, the quadratic variation of f over the interval $[0,t]$, $Q_t(f)$, is defined as*

$$Q_t(f) = \sup_{\Pi \in \mathcal{P}} \sum_{k=1}^{|\Pi|} |f(t_k) - f(t_{k-1})|^2. \tag{8}$$

Assuming that f is a continuous function, by noticing that for any $\Pi \in \mathcal{P}$

$$\sum_{k=1}^{|\Pi|} |f(t_k) - f(t_{k-1})| \geq \frac{\sum_{k=1}^{|\Pi|} |f(t_k) - f(t_{k-1})|^2}{\max_{1 \leq k \leq |\Pi|} |f(t_k) - f(t_{k-1})|}$$

we have that if $f \in BV$ then $Q_t(f) \equiv 0$ and that any function with non-zero quadratic variation has infinite total variation.

Remark 1. *In case of a stochastic process $\{X(t), t \geq 0\}$, the definitions for total variation and quadratic variation stay the same with the only remark that the limits are intended in probability sense.*

Proposition 1. *The quadratic variation function of the standard Brownian motion, $B(t)$, is given by*

$$Q_t(B) = t.$$

Proof. Given that $\mathbb{E}[[B(t_k) - B(t_{k-1})]^2] = \mathbb{V}\text{ar}[B(t_k) - B(t_{k-1})] = (t_k - t_{k-1})$ we immediately get

$$\mathbb{E}\left[\sum_{k=1}^{|\Pi|} [B(t_k) - B(t_{k-1})]^2\right] = t.$$

It is enough to prove that

$$\mathbb{V}\text{ar}\left[\sum_{k=1}^{|\Pi|} [B(t_k) - B(t_{k-1})]^2\right] \to 0 \quad \text{as } ||\Pi|| \to 0$$

to get that the convergence in (6) holds true.

We have

$$\mathbb{V}\text{ar}\left[\sum_{k=1}^{|\Pi|} [B(t_k) - B(t_{k-1})]^2\right] = \sum_{k=1}^{|\Pi|} \mathbb{V}\text{ar}\left[[B(t_k) - B(t_{k-1})]^2\right] = 2\sum_{k=1}^{|\Pi|} [t_k - t_{k-1}]^2$$

Therefore

$$\mathbb{V}\text{ar}\left[\sum_{k=1}^{|\Pi|} [B(t_k) - B(t_{k-1})]^2\right] \to Q_t(t) = 0,$$

where in the last equality we used the fact that the linear function t has finite total variation $V_t(t) = t$ and therefore for the remark above zero quadratic variation.

Noticing that the $\mathcal{L}^2$-convergence implies the convergence in probability, we get the result. □

1.4 Definition of the Itôintegral

From the previous section we have noticed that there could be some ambiguity in the definition of the stochastic integral via the limit of a Riemann sum, to avoid this we introduce the following definition.

Definition 3. *Given a standard Brownian Motion $\{B(t), t \geq 0\}$, we say that a stochastic process $\{X(t), t \geq 0\}$, is adapted to the filtration generated of the Brownian motion if*

$$1\{X(t) \in A\} = f(B(s), 0 \leq s \leq t),$$

for any event A.

The above definition just states that the process $X(t)$ cannot anticipate information about the Brownian Motion after time t. Given the definition of an adapted process we are now ready to define the Itôintegral.

Definition 4. *Given the standard Brownian Motion $\{B(t), t \geq 0\}$ and an adapted stochastic process $\{X(t), t \geq 0\}$ satisfying the condition*

$$\int_0^t \mathbb{E}[X^2(s)]ds < \infty$$

the Itôintegral, $I_t(X)$ is defined as

$$I_t(X) = \int_0^t X(s)\,dB(s) = \lim_{||\Pi|| \to 0} \sum_{k=1}^{|\Pi|} X(t_{k-1})[B(t_k) - B(t_{k-1})]$$

where the limit is meant in $\mathcal{L}^2$ sense.

1.5 The Itôintegral - Properties

Proposition 2. *The Itôintegral shares the following properties*

a) $\mathbb{E}\left[\int_0^t X(s)\,dB(s)\right] = 0$

b) $\mathbb{V}\mathrm{ar}\left[\int_0^t X(s)\,dB(s)\right] = \int_0^t \mathbb{E}[X^2(s)]\,ds$ ***(Itôisometry)***

c) $\int_0^t a_1\,X_1(s) + a_2 X_2(s)\,dB(s) = a_1 \int_0^t X_1(s)\,dB(s) + a_2 \int_0^t X_2(s)\,dB(s)$ ***(Linearity)***

d) $M(t) = M(0) + \int_0^t X(s)\,dB(s)$ *is a continuous Martingale* ***(Martingale property)***

In addition it is easy to prove that

$$\int_0^t a\,dB(s) = a\,B(t) \tag{9}$$

and

$$\int_0^t B(s)\,dB(s) = \frac{B^2(t)}{2} - \frac{t}{2}. \tag{10}$$

Definition 5. *An Itôprocess, $\{X(t), 0 \leq t \leq T\}$, is any stochastic process that may be written in the following form*

$$X(t) = X(0) + \int_0^t g(s)\,ds + \int_0^t h(s)\,dB(s) \tag{11}$$

where $g(\omega, s)$ and $h(\omega, s)$ are two adapted stochastic processes such that

$$\mathbb{P}\{\int_0^T |g(s)|\,ds < \infty\} = 1$$
$$\mathbb{P}\{\int_0^T |h(s)|^2\,ds < \infty\} = 1.$$

Equation (11) can be written in differential form in the following way

$$dX(t) = g(t)\,dt + h(t)\,dB(t) \tag{12}$$

The integration of a deterministic function with respect to the Brownian motion yields to a Gaussian process whose parameter functions are easy to compute, as it is shown in the following proposition.

Proposition 3. *Given a deterministic function $f(t)$ the Itôprocess*

$$I_t(f) = \int_0^t f(u)\,dB(u)$$

is a Gaussian process with zero mean function and covariance function

$$\mathbb{C}\text{ov}(I_t(f), I_s(f)) = \mathbb{E}\left[\int_0^t \int_0^s f(\tau)\,f(\sigma)\,dB(\sigma)dB(\tau)\right] = \int_0^{t\wedge s} f^2(u)du.$$

Proof. Use the Itôisometry and the independence of the increments of the Brownian motion. □

An important formula to compute the values of the stochastic integrals is the *Itôformula* given in the following proposition.

Proposition 4 (ItôLemma)**.** *Given an Itôprocess $\{X(t), t \geq 0\}$ and a function $f(t,x) \in \mathcal{C}^2(\mathbb{R}^+, \mathbb{R})$ then the following relation holds true*

$$f(t, X(t)) = f(0, X(0)) + \int_0^t f_t(s, X(s))\,ds + \int_0^t f_x(s, X(s))\,dX(s) + \frac{1}{2}\int_0^t f_{xx}(s, X(s))\,(dX(s))^2, \tag{13}$$

where $f_t(t,x) = \partial_t f(t,x)$, $f_x(t,x) = \partial_x f(t,x)$ and $f_{xx}(t,x) = \partial_x^2 f(t,x)$.

In the formula above $dX(s)$ is given by equation (12) while $(dX(s))^2 = h^2(s)\,ds$ with $h(t)$ being the function appearing in the definition of $X(t)$ in (11). $(dX(s))^2$ can be formally obtained by computing $dX(s) \times dX(s)$ from equation(12) and using the following resuming table

$\times$	dt	$dB(t)$
dt	*0*	*0*
$dB(t)$	*0*	dt

In differential form equation (13) is written in the following way

$$df(t, X(t)) = f_t(t, X(t))\,dt + f_x(t, X(t))\,dX(t) + \frac{1}{2} f_{xx}(t, X(t))\,(dX(t))^2. \tag{14}$$

Remark 2. *For the case $X(t)$ is the standard Brownian motion, the Itôformula (13) simplifies in*

$$f(t, B(t)) = f(0,0) + \int_0^t [f_t(s, B(s)) + \frac{1}{2} f_{xx}(s, B(s))]\,ds + \int_0^t f_x(s, B(s))\,dB(s) \tag{15}$$

and in differential form

$$df(t, B(t)) = [f_t(t, B(t)) + \frac{1}{2} f_{xx}(t, B(t))]\,dt + f_x(t, B(t))\,dB(t). \tag{16}$$

Example 1. *Using Itôformula it is easy to compute the value of $\int_0^t B(s)\,dB(s)$. Choosing $f(t,x) = x^2$, such that $f_t(t,x) = 0$, $f_x(t,x) = 2x$ and $f_{xx}(t,x) = 2$, by (15) we get*

$$B^2(t) = \int_0^t ds + \int_0^t 2\,B(s)\,dB(s)$$

and therefore

$$\int_0^t B(s)\,dB(s) = \frac{B^2(t)}{2} - \frac{t}{2}$$

that agrees with (10).

1.6 Chain Rule

The following propositions underline the differences between the stochastic integral and the classical Riemann-Stiltjes integral.

Proposition 5 (Riemann-Stiltjes' Chain Rule)**.** *Given a continuous differentiable function $f \in \mathcal{C}^1(\mathbb{R})$ and a BF function $\{x(t), t \geq 0\}$, the following relation holds true*

$$f(x(t)) = f(x(0)) + \int_0^t f'(x(s))\, dx(s).$$

Proposition 6 (Itô's Chain Rule)**.** *Given a continuous twice differentiable function $f \in \mathcal{C}^2(\mathbb{R})$ and $\{X(t), t \geq 0\}$ a function with finite quadratic variation, the following relation holds true*

$$f(X(t)) = f(X(0)) + \int_0^t f'(X(s))\, dX(s) + \frac{1}{2} \int_0^t f''(X(s))\, (dX(s))^2$$

1.7 Differential of the product

In this section we show an application of the Itôformula.

Proposition 7 (Differential of the product of functions)**.** *Consider to functions, $g(t) \in \mathcal{C}^1(\mathbb{R}^+)$ of bounded variation and $X(t)$ possibly of unbounded variation, than the following formula holds true*

$$d(g(t)\, X(t)) = X(t)\, dg(t) + g(t)\, dX(t)\ .$$

Proof. Define $f(t, x) = g(t)\, x$, with $f_t(t, x) = g'(t)\, x$, $f_x(t, x) = g(t)$ and $f_{xx}(t, x) = 0$, by applying Itô's lemma we immediately get

$$d(g(t)\, X(t)) = df(t, X(t)) = g'(t)\, X(t)\, dt + g(t)\, dX(t)\ .$$

Having that $g'(t)\, dt = dg(t)$, the result follows. □

The formula above coincides with the one of the classical differential calculus.