Notes

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1 The Renewal Equation

The renewal equation is an important integral equation that appears often in Renewal Theory. Its expression is given by

$$Z(t) = H(t) + \int_0^t Z(t-s)\,F(s), \quad t \ge 0 \tag{1}$$

with $Z(t)$, $H(t)$ and $F(t)$ general functions defined for $t \ge 0$. Usually the functions $F(t)$ and $H(t)$ are known and the task is to look for a function $Z(t)$ that satisfies equation (1) for all $t \ge 0$.

Remembering that the Stieltjes convolution $Z * F(t)$ of two functions defined on $\mathbb{R}^+$, $Z(t)$ and $F(t)$, is given by

$$Z * F(t) = \int_0^t Z(t-s)\,F(ds)$$

the renewal equation can be expressed also in the following alternative form

$$Z(t) = H(t) + Z * F(t).$$

In addition, since it is known that the Laplace transform of a convolution is given by the product of the Laplace transforms of the convoluted functions, i.e $\mathcal{L}[Z*F](s) = \mathcal{L}[Z](s) \cdot \mathcal{L}[F](s)$, equation (1) is equivalent to the following

$$\tilde{Z}(s) = \tilde{H}(s) + \tilde{Z}(s)\,\tilde{F}(s),$$

where we denoted by $\tilde{Z}(s) = \mathcal{L}[Z](s)$, $\tilde{F}(s) = \mathcal{L}[F](s)$ and $\tilde{H}(s) = \mathcal{L}[G](s)$ the Laplace transforms of $Z(t)$, $F(t)$ and $H(t)$ respectively.

Solving for $\tilde{Z}(s)$ we get that the Laplace transform of the solution function $Z(t)$ in (1) is given by

$$\tilde{Z}(s) = \frac{\tilde{H}(s)}{1 - \tilde{F}(s)}. \tag{2}$$

Since the Laplace transform uniquely characterizes the transformed function, the unique solution of the renewal equation (1) is given by the inverse transform of (2), i.e. $Z(t) = \mathcal{L}^{-1}[\tilde{Z}](t)$.

Noticing that

$$\frac{1}{1-\tilde{F}(s)} = \sum_{n=0}^{\infty} \tilde{F}(s)^n$$

we get directly that

$$Z(t) = \mathcal{L}^{-1}[\tilde{Z}](t) = \mathcal{L}^{-1}\left[\frac{\tilde{H}}{1-\tilde{F}}\right](t) = H * \sum_{n=0}^{\infty} F^{*n}(t), \tag{3}$$

where we denoted by $F^{*(n+1)}(t) = F^{*n} * F(t)$, for $n \ge 0$, having defined $F^{*0}(t) \equiv 1$.

Example 1. *Let $N(t)$ be a renewal process with general inter renewal time τ whose distribution is denoted by $F(t) = \mathbb{P}\{\tau \le t\}$. It is known that the renewal function $m(t) = \mathbb{E}[N(t)]$ can be expressed in the following form*

$$m(t) = \sum_{n=1}^{\infty} F^{*n}(t) = F * \sum_{n=0}^{\infty} F^{*n}(t),$$

therefore it is solution of the following renewal equation

$$m(t) = F(t) + m * F(t).$$

Example 2. *Let $N_D(t)$ be a delayed renewal process with general inter renewal time τ whose distribution is denoted by $F(t) = \mathbb{P}\{\tau \leq t\}$ and with $H(t)$ being the distribution of the initial delay. Then the delayed renewal function $m_D(t) = \mathbb{E}[N_D(t)]$ has the following known expression*

$$m_D(t) = H * \sum_{n=0}^{\infty} F^{*n}(t),$$

therefore it is solution of the following renewal equation

$$m_D(t) = H(t) + m_D * F(t).$$