Notes

February 21st, 2012

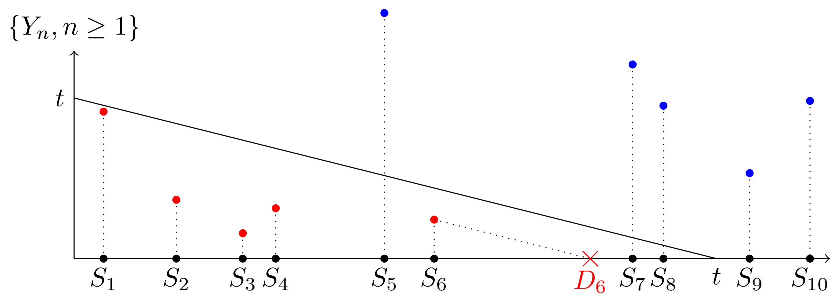
The $M/G/\infty$ Queue

In this section we study the $M/G/\infty$ queue. It assumes that at a service station a flow of customers arrives according to a Poisson process, that is the point S_n on the real line denotes the arrival time of the n-th customer. Each customer requires a service time, that we assume is given by a random variable, say Y_n, that is independent of th sequence of arrival times and from the other service time r.v. as well. The distribution of the variable Y_n is general, that explains the second symbol in the name of the model, and we denote it by $G(y) = \mathbb{P}\{Y_n \leq y)$. When a customer enters the service station, s/he immediately starts to be served because the number of server is assumed to be infinite, that is the third symbol in the name of the model. The assumption that the number of server is infinite makes this system easy to study in some of its characteristics as the processing of any customer in the server station does not at all affects the processing of other customers already present or arriving later to the station.

One possible applications of this model is car traffic flowing through a highway. If we assume that cars access the highway independently and with regular frequency, we can think that in low traffic condition their traveling times will not being affected by the presence of the other cars, justifying the additional assumption that they are independent and identically distributed random variables modeling the variability in the driving speeds of the cars. The customers still in service will be the cars still traversing the highway while the served customers model the cars that already crossed all the highway.

Let call by $N(t)$ the number of customers arrived to the queue up to time t, that is $N(t)$ is a λ homogeneous Poisson Process. We then define by $Q(t)$ the number of customer in the queue at time t and by $D(t)$ the ones that already left the system.

It is trivial to note that $N(t) = Q(t) + D(t)$. Since the departure time of the n-th customer, let us say $D_n = S_n + Y_n$, we have that s/he is still in the queue at time t if $D_n > t$ or equivalently if $Y_n > t - S_n$. We can plot this situation in the following way; the black points are the arrival times, and the blue/red points are at a vertical distance from the time-axis equal to the service time of the corresponding customer. If this points is above to the line $f(s) = s - t$ then the customer is still in the system at time t, and the point is blue color, otherwise s/he already left the system and the dot is colored red.



Proceeding as in Proposition 4 of [1], we can show that $Q(t)$ and $D(t)$ are Poisson random variables as it shows the following proposition.

Proposition 1. *Given an $M/M/\infty$ queue where the arrival process has rate λ and the service times are independent and have common distribution G. Then at a fixed time $t \geq 0$ the number of customers in the system and the number of customers that left the system in this interval are independent and distributed according a Poisson distribution with parameters respectively $\lambda^Q(t) = \frac{\lambda}{t}\int_0^t \bar{G}(t-s)\,ds$ and $\lambda^D(t) = \frac{\lambda}{t}\int_0^t G(t-s)\,ds$.*

$$\begin{aligned}
\mathbb{P}(D(t) = n,\, Q(t) = m) &= \mathbb{P}(D(t) = n | N(t) = n+m)\,\mathbb{P}(N(t) = n+m) \\
&= \mathbb{P}(\sum_{i=1}^{n+m} 1\{Y_i \leq t - S_i)\} = n | N(t) = n+m)\,\frac{(\lambda\,t)^{n+m}}{n+m!} e^{-\lambda\,t} \\
&= \mathbb{P}(\sum_{i=1}^{n+m} 1\{Y_i \leq t - U_{(i)}\} = n)\,\frac{(\lambda\,t)^{n+m}}{n+m!} e^{-\lambda\,t} \\
&\stackrel{\mathcal{L}}{=} \mathbb{P}(\sum_{i=1}^{n+m} 1\{Y_i \leq t - U_i\} = n)\,\frac{(\lambda\,t)^{n+m}}{n+m!} e^{-\lambda\,t}
\end{aligned}$$

In the third equality we used Proposition 3 of [1]. The random variables $1\{Y_i \le t - U_i\}$, $1 \le i \le n+m$, are i.i.d. Bernoulli with parameter

$$p = \int_0^t \mathbb{P}\{Y_i \le t - u\}\, dF_{U_i}(u) = \frac{1}{t}\int_0^t G(t-u)\, du$$

so it follows that the random variable $Z = \sum_{i=1}^{n+m} 1\{Y_i \le t - U_i\} \sim \mathrm{Bin}(n+m, p)$. Substituting in (1) we finally get

$$\begin{aligned}
\mathbb{P}(S(t) = n,\, Q(t) = m) &= \mathbb{P}(Z = n)\, \frac{(\lambda\, t)^n}{n!} e^{-\lambda\, t} = \binom{n+m}{n} p^n\, q^m\, \frac{(\lambda\, t)^{n+m}}{n+m!} e^{-\lambda\, t} \\
&= \frac{n+m!}{n!\, m!}\, p^n\, q^m\, \frac{(\lambda\, t)^{n+m}}{n+m!} e^{-\lambda (p+q)\, t} \\
&= \frac{(\lambda\, p\, t)^n}{n!}\, e^{-\lambda\, p\, t}\, \frac{(\lambda\, q\, t)^m}{m!}\, e^{-\lambda\, q\, t} \qquad (1)
\end{aligned}$$

with $q = 1 - p$ that gives the result.

Exercises

1. Given an $M/G/\infty$ queue with arrival rate λ and service distribution G, show that the departure process $D(t)$ is Poisson process.

References

[1] B.D'Auria (2012): *Stochastic processes - Notes of February 16, 2012.*