Modeling Risk: The heartbreak of asymptotic independence

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1. Introduction: Finding Hidden Risks

1.1. Background.

Suppose

$$\boldsymbol{X} = (X_1, \dots, X_d)$$

is a *risk vector*. Imagine X_i is

- loss from ith asset in portfolio;
- concentration of ith pollutant;
- car maker's warranty exposure over a month for ith car model in lineup.

Goal: Estimate the probability of a *risk region* $\mathcal{R}$

$$P[\boldsymbol{X} \in \mathcal{R}]$$

where $\mathcal{R}$ is beyond the range of observed data.

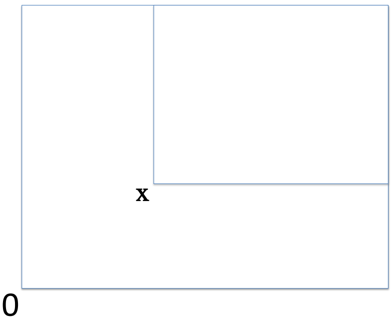
Example:

$d = 2$ and

$\mathcal{R} = (\mathbf{x}, \boldsymbol{\infty}] = (x_1, \infty] \times (x_2, \infty]$

and

$P[\boldsymbol{X} \in \mathcal{R}] = P[X_1 > x_1, X_2 > x_2].$



Risk contagion: Can two or more components of the risk vector $\boldsymbol{X}$ be simultaneously large? Typically,

large=beyond the range of the data.

2. Mathematical foundation: Regularly varying functions and measures

2.1. Regularly varying functions

A function $U : \mathbb{R}_+^d \mapsto \mathbb{R}_+$ is multivariate regularly varying if

$$\lim_{t\to\infty} \frac{U(t\mathbf{x})}{U(t\mathbf{1})} = \lambda(\mathbf{x}) \neq 0,$$

for $\mathbf{x} \geq \mathbf{0}$, $\mathbf{x} \neq \mathbf{0}$.

- If $d = 1$, limit must be a power function and we are dealing with functions U which are asymptotically like power functions; for $d = 1$,

  $$\frac{U(tx)}{U(t)} \to x^\rho, \quad \rho \in \mathbb{R}.$$

  Call ρ the index and when $d = 1$ we write $U \in RV_\rho$.

- When $d > 1$, a scaling argument shows $\exists\ \rho \in \mathbb{R}$ and

  $$\lambda(t\mathbf{x}) = t^\rho \lambda(\mathbf{x}),$$

  and $U(t\mathbf{1}) \in RV_\rho$.

- Therefore, equivalent formulation is there exists $V \in RV_\rho$ such that
$$\frac{U(t\mathbf{x})}{V(t)} \to \lambda(\mathbf{x}) \neq 0.$$
- Possible choice: $V(t) = U(t\mathbf{1})$.
- Sequential version when $\rho > 0$): $\exists\, b_n \to \infty$ such that
$$\frac{U(b_n\mathbf{x})}{n} \to \lambda(\mathbf{x}).$$
- Can set $b_n = V^{\leftarrow}(n)$.

2.2. Connection to domain of attraction (DOA) characterizations.

Suppose $\{X_n, n \geq 1\}$ are iid non-negative, common distribution function $F(x)$. The extreme is

$$M_n = \bigvee_{i=1}^{n} X_i = \max\{X_1, \dots, X_n\}.$$

One of the extreme value distributions is the Frechet:

$$\Phi_\alpha(x) := \exp\{-x^{-\alpha}\}, \quad x > 0,\ \alpha > 0.$$

Questions:

- What are conditions on F, called *domain of attraction conditions*, so that there exists $b_n > 0$ such that

  $$P[b_n^{-1} M_n \leq x] = F^n(b_n x) \to \Phi_\alpha(x). \qquad \text{(DOA-frechet)}$$

  When (DOA-frechet) holds, we say F is in the domain of attraction of Φ_α and write $F \in D(\Phi_\alpha)$.

- How do you characterize the normalization sequence $\{b_n\}$?

Answers:

- DofA: One argues that we must have

$$x_0 = \sup\{x : F(x) < 1\} = \infty,$$

and furthermore

$$b_n \to \infty.$$

In (DOA-frechet), take logarithms to get for

$$\lim_{n\to\infty} n(-\log F(b_n x)) = x^{-\alpha}, \quad x > 0.$$

Use

$$-\log(1-z) \sim z, \quad (z \to 0,$$

and (DOA-frechet) is equivalent to

$$\lim_{n\to\infty} n(1 - F(b_n x)) = x^{-\alpha}, \quad x > 0. \tag{1}$$

This is the sequential version of regular variation for $\bar{F} := 1 - F$.

- Characterize b_n: Set $U(x) = 1/(1 - F(x))$ and (1) is the same as

$$U(b_n x)/n \to x^{\alpha}, \quad x > 0,$$

and inverting, we find that

$$\frac{U^{\leftarrow}(ny)}{b_n} \to y^{1/\alpha}, \quad y > 0. \tag{2}$$

Conclude:

$$U^{\leftarrow}(n) = (1/(1-F))^{\leftarrow}(n) = F^{\leftarrow}(1-\frac{1}{n}) \sim b_n$$

and this determines b_n (convergence to types theorem).

2.2.1. Summary: Connecting regular variation and domains of attraction in one dimension.

With

$$\Phi_\alpha(x) = \exp\{-x^{-\alpha}\}, \quad x > 0, \ \alpha > 0,$$

we have

$$F \in D(\Phi_\alpha) \text{ iff } \lim_{t\to\infty} \frac{\bar{F}(tx)}{\bar{F}(t)} = x^{-\alpha}, \ x > 0;$$

that is, $\bar{F} \in RV_{-\alpha}$.

2.3. Behavior of one dimensional regularly varying functions:

Regularly varying functions behave asymptotically like power functions. Helpful notation: Call $L(x)$ slowly varying if $L(\cdot) \in RV_0$. Then if

$$U \in RV_\rho$$

we have

$$L(x) := U(x)/x^\rho \in RV_0$$

and we can write

$$U(x) = x^\rho L(x).$$

Rules for manipulating:

- Karamata theorem: For $\rho > -1$,

$$\int_0^x U(t)dt$$

behaves as if $L(t)$ comes out of the integral and the power part integrates. So if $U(x) = x^\rho L(x)$, then

$$\int_0^x U(t)dt = \int_0^x t^\rho L(t)dt$$

$$\sim L(x) \int_0^x t^\rho dt = L(x) \frac{x^{\rho+1}}{\rho+1}$$

$$= \frac{xU(x)}{\rho+1}.$$

- Differentiation: If $U \in RV_\rho$ has a monotone density $u(x)$, then $u(x) \in RV_{\rho-1}$ (as if it were a power function).
- Regularly varying functions have smooth asymptotically equivalent versions which comes from the *Karamata representation*: if $U \in RV_\rho$,

$$U(x) = c(x) \exp\{\int_1^x \frac{\rho(s)}{s} ds\},$$

where

$$c(x) \to c_0, \quad \rho(t) \to \rho.$$

So

$$U(x) \sim c_0 \exp\{\int_1^x \frac{\rho(s)}{s} ds\},$$

and the right side can be made as smooth as one likes (eg, infinitely differentiable).

- The regular variation ratio converges locally uniformly; eg, if $U \in RV_\rho$,

$$\lim_{t\to\infty} \frac{U(tx)}{U(t)} = x^\rho,$$

uniformly on $[a,b]$, $0 < a < b < \infty$.

  - If $\rho < 0$, uniform convergence on $[b, \infty)$, $b > 0$.

- Inversion: If $U \in RV_\rho$, $\rho > 0$, regular variation $U \in RV_\rho$ implies $U^\leftarrow \in RV_{1/\rho}$:

$$\lim_{t\to\infty} \frac{U^\leftarrow(tx)}{U^\leftarrow(t)} = x^{1/\rho}.$$

2.4. Multivariate Regular Variation for Multivariate Distribution Functions and Measures on $\mathbb{R}_+^d$

Application to distributions: Let $\boldsymbol{Z}$ be a random vector in $\mathbb{R}_+^d$ with df F. A *regularly varying tail* means

$$\frac{1 - F(t\mathbf{x})}{1 - F(t\mathbf{1})} \to \nu_*([\mathbf{0}, \mathbf{x}]^c),$$

for some Radon measure ν_*. Awkward to deal with multivariate df's and better to deal with measures.
Let

$$\begin{aligned}
\mathbb{E} =& [0, \infty]^d \setminus \{\mathbf{0}\} \\
\aleph =& \{\mathbf{x} \in \mathbb{E} : \|\mathbf{x}\| = 1\} \qquad \text{(unit sphere)}, \\
R =& \|\boldsymbol{Z}\|, \quad \boldsymbol{\Theta} = \frac{\boldsymbol{Z}}{\|\boldsymbol{Z}\|} \in \aleph \qquad \text{(polar coordinates)}.
\end{aligned}$$

The following are equivalent.

1. $\exists$ a Radon measure ν_* on $\mathbb{E}$ such that

$$\begin{aligned}
\lim_{t\to\infty} \frac{1 - F(t\mathbf{x})}{1 - F(t\mathbf{1})} =& \lim_{t\to\infty} \frac{\mathbb{P}\left[\frac{\boldsymbol{Z}_1}{t} \in [\mathbf{0}, \mathbf{x}]^c\right]}{\mathbb{P}\left[\frac{\boldsymbol{Z}_1}{t} \in [\mathbf{0}, \mathbf{1}]^c\right]} \\
=& c\nu_*\left([\mathbf{0}, \mathbf{x}]^c\right),
\end{aligned}$$

some $c > 0$ and for all points $\mathbf{x} \in [\mathbf{0}, \infty) \setminus \{\mathbf{0}\}$ which are continuity points of $\nu_*([\mathbf{0}, \cdot]^c)$.

2. $\exists$ a function $b(t) \to \infty$ and a Radon measure ν_* on $\mathbb{E}$ such that in $M_+(\mathbb{E})$

$$t\mathbb{P}\big[\frac{\boldsymbol{Z}}{b(t)} \in \cdot\big] \xrightarrow{v} \nu_*, \quad t \to \infty.$$

3. $\exists$ a pm $S(\cdot)$ on $\aleph$ and $b(t) \to \infty$ such that

$$t\mathbb{P}[(\frac{R}{b(t)}, \boldsymbol{\Theta}) \in \cdot] \xrightarrow{v} c\nu_\alpha \times S$$

in $M_+\big(((0, \infty] \times \aleph\big)$, where $c > 0$.

Notes:

- Can replace function $b(t)$ by sequence $b(n)$.
- $\xrightarrow{v}$ means vague convergence defined as follows: Let $M_+(\mathbb{E})$ be the Radon measures on $\mathbb{E}$. (Radon means the measure is finite on relatively compact sets.) $M_+(\mathbb{E})$ can be metrized by vague convergence: Let $\mu_n(\cdot), n \geq 0$ be measures in $M_+(\mathbb{E})$. Then

$$\mu_n \xrightarrow{v} \mu_0$$

iff

$$\mu_n(f) := \int_E f d\mu_n \to \int_E f d\mu =: \mu(f) \quad (n \to \infty)$$

for all non-negative, continuous functions with compact support on $\mathbb{E}$.

- Generally, vague convergence can be reduced to convergence of measures on a class of rectangles suited to the compact sets of $\mathbb{E}$.

These conditions imply:

for any sequence $k = k(n) \to \infty$ such that $n/k \to \infty$ we have

4. In $M_+(\mathbb{E})$,

$$\frac{1}{k}\sum_{i=1}^{n} \epsilon_{\boldsymbol{Z}_i/b(\frac{n}{k})} \Rightarrow \nu \qquad (*)$$

$$\frac{1}{k}\sum_{i=1}^{n} \epsilon_{(R_i/b(n/k),\Theta_i)} \Rightarrow (c\nu_\alpha \times S). \qquad (**)$$

and (4) is equivalent to any of (1)–(3), provided $k(\cdot)$ satisfies $k(n) \sim k(n+1)$.

Ignore fact $b(\cdot)$ unknown:
$\to$ LHS of Eqn (*) is a consistent estimator of ν.
$\to$ From (**), consistent estimator of S is

$$\frac{\sum_{i=1}^{n} \epsilon_{(R_i/b(n/k),\Theta_i)}[1,\infty] \times \cdot)}{\sum_{i=1}^{n} \epsilon_{R_i/b(n/k)}[1,\infty]}.$$

We need the following:

Let $\mathbb{C}$ be a closed cone in $\mathbb{R}_+^d$; that is,

$$\mathbf{x} \in \mathbb{C} \Rightarrow t\mathbf{x} \in \mathbb{C}, \quad \forall t > 0.$$

The random vector $\boldsymbol{Z} \in \mathbb{R}_+^d$ has a distribution tail which is regularly varying on $\mathbb{C}$ with limit measure $\nu_{\mathbb{C}}(\cdot)$ if for nice sets $A \subset \mathbb{C}$ we have

$$tP[\frac{\boldsymbol{Z}}{t} \in A] \to v_{\mathbb{C}}(A).$$

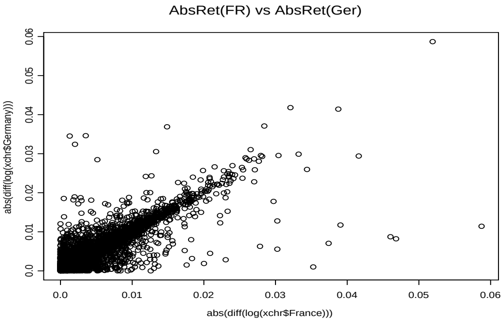
Example: $\mathbb{C} = (\mathbf{0}, \boldsymbol{\infty}]$.

2.5. Statistical difficulty.

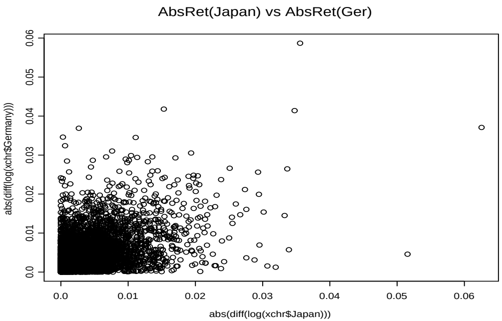
- This formulation is good for theory but bad for applications.
- Being able to norm each component by the same $b(t)$ means marginal tails are the same–almost never happens in practice. Multivariate data from a distribution with heavy tailed marginals, never have the same α's.
  - Set
$$\boldsymbol{Z} = (Z^{(1)}, \dots, Z^{(d)}).$$
    Norming each component with the same $b(t)$ means
$$\mathbb{P}[Z^{(i)} > x] \sim c_{ij}\mathbb{P}[Z^{(j)} > x], \quad x \to \infty.$$
    and if $c_{ij} > 0$, then the tail index of $Z^{(i)}$ and $Z^{(j)}$ are the same.
- Not true in practice. And what is the dependence structure?
- Examples:
  - Absolute returns Xchr vs USD of (FR, JAP).
  - (Size of document downloaded, download time); etc.

Absolute log-returns forex 2 Eurpean currencies (prior to Euro)

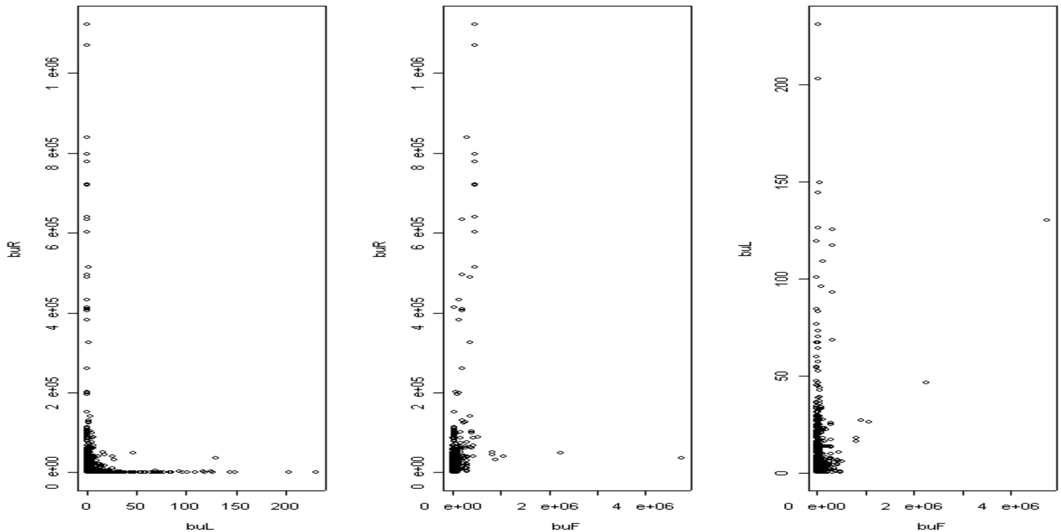
- Daily absolute log-returns of French franc vs German mark against the USD.
- Jan 1971 – Feb 1994.
- 6041 days.



- Daily absolute log-returns of the Japanese yen vs the German mark.
- Jan 1971 – Feb 1994.
- 6041 days.



Boston Univ http request data: size, download rate, duration.



BU: duration vs rate (l); size vs rate (c); size vs duration (r).

α	$\hat{\alpha}_{\text{size}}$	$\hat{\alpha}_{\text{rate}}$	$\hat{\alpha}_{\text{duration}}$
estimated value	1.15	1.13	1.4

Table 1: Tail parameter estimates.

2.5.1. More flexibility.

A more flexible defn of a multivariate heavy tail is: $\exists\, b_n^{(i)} \to \infty$ for $i = 1, \dots, d$ and $\exists$ Radon ν such that

$$n\mathbb{P}\Big[\Big(\frac{Z^{(i)}}{b_n^{(i)}}, i = 1, \dots, d\Big) \in \cdot\ \Big] \to \nu(\cdot). \tag{3}$$

Theorem. Suppose $\boldsymbol{Z} \sim F$ and $Z^{(i)} \sim F_{(i)}$. If (3) and

$$n\mathbb{P}\Big[\frac{Z^{(i)}}{b_n^{(i)}} \in \cdot\ \Big] \to \nu_{\alpha_i}(\cdot), \quad \nu_{\alpha_i}(x, \infty] = x^{-\alpha_i},\ x > 0, \alpha_i > 0,\ \forall i,$$

then

$$nF_*(n\cdot\,) = n\mathbb{P}\left[\left(\frac{\frac{1}{1-F_{(i)}(Z^{(i)})}}{n}, i = 1, \dots, d\right) \in \cdot\ \right] \to \nu_*(\cdot)$$

where ν_* is *standard*; that is, Radon and

$$\nu_*(tA) = t^{-1}\nu_*(A).$$

Note if for $i = 1, \dots, d$

$$1 - F_{(i)}(x) \sim x^{-\alpha_i}, \quad x \to \infty,$$

then

$$n\mathbb{P}\Big[\Big(\frac{(Z^{(i)})^{\alpha_i}}{n},\ i = 1, \dots, d\Big) \in \cdot\ \Big] \to \nu_*(\cdot).$$

2.5.2. How to get the standard case in practice:

(a) Simple minded. Hope (assume, pray) $1 - F_{(i)}(x) \sim x^{-\alpha_i}$ for all i and then power up. (Slow variation hard (impossible?) to detect in practice and this works reasonably.)

BUT: Must estimate α's. (Ouch!)

(b) Use ranks method.

BUT: Lose independence among observations.

2.6. Significance of the limit measure.

The limit measure ν_* controls the (asymptotic) dependence structure.

2.6.1. Asymptotic independence.

The distribution F of $\boldsymbol{Z}$ possesses the property of *asymptotic independence* if

1. $\nu_*\big((\mathbf{0}, \boldsymbol{\infty})\big) = 0$ so that ν_* concentrates on the axes;

   OR

2. S concentrates on $\{\boldsymbol{e}_i, i = 1, \dots, d\}$.

For $d = 2$ this means

$$\mathbb{P}[Z^{(2)} > t | Z^{(1)} > t] = \frac{\mathbb{P}[Z^{(2)} > t, Z^{(1)} > t]}{\mathbb{P}[Z^{(1)} > t]}$$

$$\to (const)\nu_*\Big(\mathbf{1}, \boldsymbol{\infty}\Big] = 0.$$

Remarks:

- The probability of both components being large is negligible.
- The concept was invented to answer the following query: Let $\{\boldsymbol{Z}_j, j \geq 1\}$ be iid, non-negative random vectors in $\mathbb{R}_+^d$ with common distribution F. The necessary and sufficient condition for there to exist $b(n) \to \infty$ such that

$$P[\bigvee_{i=1}^{n} \frac{Z_i^{(l)}}{b(n)} \leq x^{(l)}, l = 1, \dots, d] \to G(\mathbf{x})$$

is $1 - F$ is multivariate regularly varying with the original definition. The necessary and sufficient condition for G to be a product distribution is asymptotic independence.

2.6.2. Asymptotic dependence.

The distribution F of $\boldsymbol{Z}_1$ possesses the property of *asymptotic dependence* if

1. ν_* concentrates on $\{t\frac{\mathbf{1}}{\|\mathbf{1}\|} : t > 0\}$, the diagonal line,

   or

2. S concentrates on $\{\frac{\mathbf{1}}{\|\mathbf{1}\|}\}$.

Means:

- $$\mathbb{P}[Z^{(2)} > t | Z^{(1)} > t] \to 1.$$

- $$G(\mathbf{x}) = G(\wedge_{i=1}^{d} x^{(i)}),$$

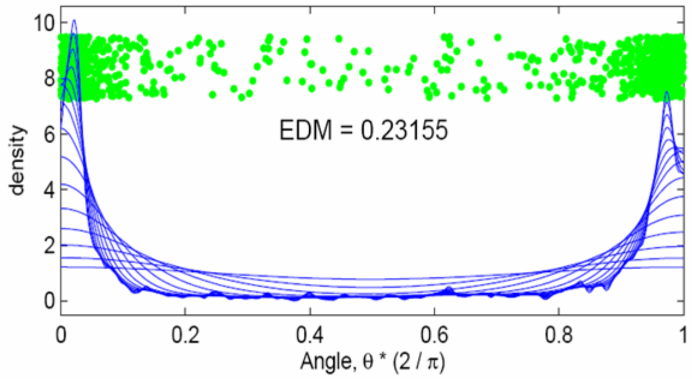
  the distribution of a random vector all of whose componenets are equal.

2.7. Examples

2.7.1. UNC

UNC HTTP response data for (size, rate) for Wednesday afternoons, 1-5:00 pm, April 2001.
Steps:

- Transform (size,rate) data using rank method.
- Convert to polar coordinates.
- Keep 2000 pairs with biggest radius vector.
- Compute density estimate for angular measure S.


EDM = 0.23155
density
Angle, θ * (2 / π)
10
8
6
4
2
0
0
0.2
0.4
0.6
0.8
1

2.7.2. The Auckland-II trace

(http://pma.nlanr.net/traces/long/auck2.html) a collection of long GPS-synchronized IP header traces captured at the University of Auckland Internet uplink since 1999.

- Connection level data characterized by packet headers.
- connection = 5 tuple (source ip, dest ip, source port, dest port, protocol)
- Amalgamate the data into clusters in hopes that cluster heads are approx Poisson.
  - No unique way to do this.
  - One proposal:
    - Organize packets into e2e streams–packets with same ip source and destination. Con: ignores application.
    - Create sessions by clustering e2e streams according to the rule that 2 consecutive packets part of the same cluster if time separation between them is below a threshold.
    - For a session, compute total payload (S), duration (D) and then (average) rate (R)

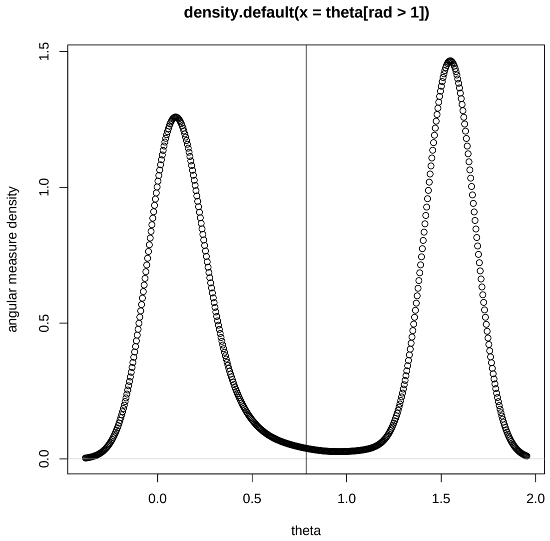


Figure 1: k=1500, Angular measure S,R; length=54343

3. General EV marginals

3.1. Asymptotic method for risk estimation

- As in one dimension, in d-dimensions, estimating probabilities of risk regions beyond the range of the data requires an assumption that enables extrapolation.
- Preparation: Interpret operations on vectors componentwise: For $\boldsymbol{x}$ and $\boldsymbol{y}$ vectors in $\mathbb{R}^d$:
  - $\mathbf{x} \vee \boldsymbol{y} = (x_i \vee y_i,\ i = 1, \dots, d)$.
  - $\mathbf{x} \wedge \boldsymbol{y} = (x_i \wedge y_i,\ i = 1, \dots, d)$.
  - $\mathbf{x} + \boldsymbol{y} = (x_i + y_i,\ i = 1, \dots, d)$.
  - $\mathbf{x}^{\boldsymbol{y}} = (x_i^{y_i},\ i = 1, \dots, d)$.
  - And for vectors $\boldsymbol{x}(j)$, $j = 1, \dots, n$

$$\bigvee_{m=1}^{n} \mathbf{x}(m) = \Big(\bigvee_{m=1}^{n} x_1(m), \dots, \bigvee_{m=1}^{n} x_d(m)\Big).$$

Intro Risks
Regular variation
EV marginals
Asy Indep
Directions
HRV
HDA
General
Final

- Usual assumption to allow extrapolation beyond the range of the data: $\boldsymbol{X}$ is in the (multivariate) domain of attraction (DOA) of an extreme value distribution; ie, $\exists\, a_i(n) > 0, b_i(n) \in \mathbb{R}$, $i = 1, \dots, d; n \geq 1$ such that if $\{\boldsymbol{X}(m), m \geq 1\}$ are iid copies of $\boldsymbol{X}$, then

$$P\Big[\bigvee_{m=1}^{n} \frac{\boldsymbol{X}(m) - \boldsymbol{b}(n)}{\boldsymbol{a}(n)} \leq \mathbf{x}\Big] = \Big(P\Big[\frac{X_i - b_i(n)}{a_i(n)} \leq x_i,\ i = 1, \dots, d\Big]\Big)^n \to G(\mathbf{x})$$

where G is a multivariate EV distribution *with non-degenerate marginals*. Equivalently, take logs, use $-\log z \sim (1 - z)$ as $z \to 1$ and

$$n\Big(1 - P\Big[\frac{X_i - b_i(n)}{a_i(n)} \leq x_i,\ i = 1, \dots, d\Big]\Big) \to -\log G(\mathbf{x})$$

and this extends to a probability statement about more general sets than just $(-\infty, \mathbf{x}]^c$:

$$nP\Big[\frac{\boldsymbol{X} - \boldsymbol{b}(n)}{\boldsymbol{a}(n)} \in \cdot\,\Big] \to \nu(\cdot) \qquad \text{(DOA)}$$

where

$$\nu(-\infty, \mathbf{x}]^c = -\log G(\mathbf{x}).$$

- Tip of the iceberg: If one is determined to use asymptotic methods and $\mathcal{R}$ is the risk region, (DOA) yields a method to estimate the risk probability:

$$P[\boldsymbol{X} \in \mathcal{R}] = P\Big[\frac{\boldsymbol{X} - \boldsymbol{b}(n)}{\boldsymbol{a}(n)} \in \frac{\mathcal{R} - \boldsymbol{b}(n)}{\boldsymbol{a}(n)}\Big] \approx \frac{1}{n}\hat{\nu}\Big(\frac{\mathcal{R} - \hat{\boldsymbol{b}}}{\hat{\boldsymbol{a}}}\Big).$$

Note: need $\frac{\mathcal{R}-\boldsymbol{b}(n)}{\boldsymbol{a}(n)}$ to be relatively compact for the asymptotics to work.

- The multivariate condition (DOA) implies one dimensional marginals are in a univariate doa. There is a *standardized* version of (DOA) which expresses the condition as multivariate regular variation on $\mathbb{E} := [\mathbf{0}, \boldsymbol{\infty}] \setminus \{\mathbf{0}\}$: Set

$$U_i(x) = \frac{1}{P[X_i > x]}$$

and

$$\boldsymbol{X}^* = \big(U_i(X_i), i = 1, \dots, d\big).$$

Then marginal convergence in (DOA) to non-degenerate EV distributions plus (DOA) is equivalent to

$$nP[\frac{\boldsymbol{X}^*}{n} \in \cdot\,] \to \nu^*(\cdot) \qquad \text{(StandRegVarE)}$$

on $\mathbb{E} = [\mathbf{0}, \boldsymbol{\infty}] \setminus \{\mathbf{0}\}$ where for $t > 0$,

$$\nu^*(t\cdot) = t^{-1}\nu^*(\cdot).$$

This is just transformation to Pareto scale.

Why this works: Take (DOA) and examine the ith marginal convergence:

$$n(1-P[X_i - b_i(n) \le a_i(n)x_i]) \to -\log G_i(x_i), \quad (G_i = ith \text{ marginal of } G),$$

or

$$n/U_i(a_i(n)x_i + b_i(n)) \to -\log G_i(x_i)$$

or

$$\frac{U_i(a_i(n)x_i + b_i(n))}{n} \to \frac{1}{-\log G_i(x_i)}.$$

Take inverses: For $y > 0$,

$$\frac{U_i^{\leftarrow}(ny) - b_i(n)}{a_i(n)} \to \Big(\frac{1}{-\log G_i}\Big)^{\leftarrow}(y) =: \psi_i(y).$$

So for $\boldsymbol{y} > \mathbf{0}$, if $\boldsymbol{X}$ satisfies (DOA),

$$
\begin{aligned}
nP([\frac{\boldsymbol{X}^*}{n} \leq \boldsymbol{y}]^c) =& nP([U_i(X_i) \leq ny_i,\ i = 1, \dots, d]^c) \\
=& nP([X_i \leq U_i^{\leftarrow}(ny_i),\ i = 1, \dots, d]^c) \\
=& nP([\frac{X_i - b_i(n)}{a_i(n)} \leq \frac{U_i^{\leftarrow}(ny_i) - b_i(n)}{a_i(n)},\ i = 1, \dots, n]^c) \\
\to& \nu\Big((-\boldsymbol{\infty}, (\psi_i(y_i),\ i = 1, \dots, d)]^c\Big) \\
=& \nu^*\big((\mathbf{0}, \boldsymbol{y}]^c\big).
\end{aligned}
$$

From this we get

$$nP[\frac{\boldsymbol{X}^*}{n} \in A] \to \nu^*(A)$$

for nice sets A.

Note since

$$\nu(-\boldsymbol{\infty}, \mathbf{x}]^c) = -\log G(\mathbf{x}),$$

we have that typically ν (and hence ν^* is an measure with infinite mass.

3.1.1. Summary.

(DOA) can be standardized to become (StandRegVarE) using marginal (copula-like) transformations.

4. Curse of asymptotic independence

- If in (DOA), the limit G is a product

$$G(\mathbf{x}) = \prod_{i=1}^{d} G_i(x_i), \qquad \text{(AsyIndep)}$$

we say $\boldsymbol{X}$ possesses *asymptotic independence.*

- Reason for the name *asymptotic independence*: The limit is a product. But $\boldsymbol{X}$ may not appear very independent (see example below). Warning: in many ways and for many people this name is an unfortunate choice.

- Unintended consequence of asymptotic independence: (AsyIndep) $\Rightarrow$

$$\nu(\{\mathbf{x} : x_i > y_i(0), x_y > y_j(0)\}) = 0,$$

for all $1 \leq i < j \leq d$ and thus such a model has no risk contagion since we estimate

$$P[\text{ two or more components of } \boldsymbol{X} \text{ are large simultaneously }] \approx 0.$$

Reason: For $d = 2$, suppose X_1, X_2 are asymptotically independent so

$$nP([X_1 \le a_1(n)x_1 + b_1(n), X_2 \le a_2(n)x_2 + b_2(n)]^c) \to -\log G_1(x_1) + -\log G_2(x_2). \quad (4)$$

Also,

$$\begin{cases} nP([X_1 \le a_1(n)x_1 + b_1(n)]^c) \to -\log G_1(x_1), \\ nP([X_1 \le a_2(n)x_2 + b_2(n)]^c) \to -\log G_2(x_2). \end{cases} \quad (5)$$

Subtract: (4)-(5), do a little set algebra and conclude

$$nP([X_1 > a_1(n)x_1 + b_1(n), X_2 > a_2(n)x_2 + b_2(n)]^c) \to 0.$$

- In standardized form: (StandRegVarE)+(AsyIndep) mean when $d = 2$,

$$\nu^*(\mathbb{E}_0) = \nu^*\big((\mathbf{0}, \boldsymbol{\infty}]\big) = 0,$$

and ν^* concentrates on the positive axes through 0.

- What to do?

4.1. How common is (AsyIndep)?

Recall data examples for (size, rate). Also:

- For $d = 2$: If $\boldsymbol{X}$ satisfies (DOA) and $X_1 \perp\!\!\!\perp X_2$ then $\boldsymbol{X}$ possesses (AsyIndep).
- If $\boldsymbol{X} = (X_1, \dots, X_d)$ is Gaussian with

  $$\mathrm{corr}(X_i, X_j) = \rho(i,j) < 1,$$

  then $\boldsymbol{X}$ possesses (AsyIndep) (Sibuya, 1960). Here the marginals of $\boldsymbol{X}$ are Gaussian and

  $$G(\mathbf{x}) = \prod_{i=1}^{d} \exp\{-e^{-x_i}\}.$$

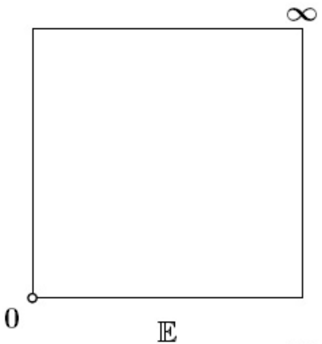
- So using the Gaussian dependence copula means you are exposed to (AsyIndep) and lack of risk contagion.
- Let $U \sim U(0,1)$ and define

  $$\boldsymbol{X} = \Big(\frac{1}{U}, \frac{1}{1-U}\Big).$$

  Since $1/U$ and $1/(1-U)$ cannot be simultaneously large, $\boldsymbol{X}$ possesses (AsyIndep). The marginals of $\boldsymbol{X}$ are Pareto and

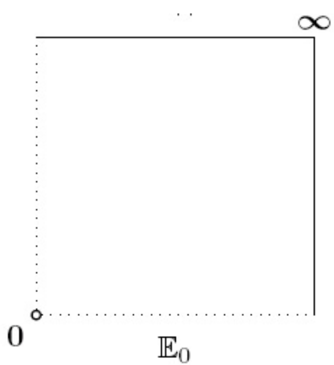
  $$G(\mathbf{x}) = \exp\{-(x_1^{-1} + x_2^{-1})\}, \qquad \mathbf{x} > \mathbf{0}.$$

5. Strategy and Directions



(AsyIndep) + (StandRegVarE) implies the limit measure $\nu^*(\cdot)$ in Pareto scale concentrates on the axes through $\mathbf{0}$.

Hint: Consider the complement of the support of ν^* and seek a lower order regular variation on this new set.



Since ν^* concentrates on axes and puts zero mass on interior of quadrant, seek (hidden) regular variation on the interior $\mathbb{E}_0 = (\mathbf{0}, \boldsymbol{\infty}]$ with index < 1. This would allow non-zero estimate of

$$P[\boldsymbol{X} > \mathbf{x}].$$

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Example. For $d=2$: If $\boldsymbol{X}=(X_1,X_2)$ and $X_1 \perp\!\!\!\perp X_2$, X_1,X_2 iid with

$$P[X_i>y]=y^{-1}, \quad y>1.$$

Then for $x_1>0$, $x_2>0$, as $n\to\infty$

$$nP[X_i>nx_i]\to x_i^{-1}, \qquad i=1,2,$$
$$nP[X_1>nx_1,X_2>nx_2]=\Big(nP[X_1>nx_1]\Big)P[X_2>nx_2]\to 0,$$

so $\boldsymbol{X}$ is regularly varying on $\mathbb{E}$ with index 1 and limit measure concentrating on the axes, and

$$nP[X_1>\sqrt{n}x_1,X_2>\sqrt{n}x_2]\to\frac{1}{x_1x_2}, \qquad x_1>0,x_2>0,$$

so $\boldsymbol{X}$ is regularly varying on $\mathbb{E}_0$ with index 2 and limit measure giving positive mass to $(\mathbf{x},\boldsymbol{\infty}]$.

Conclude for this example:

- $\boldsymbol{X}$ is regularly varying on $\mathbb{E}=[\mathbf{0},\boldsymbol{\infty}]\setminus\{\mathbf{0}\}$ with index 1 and limit measure concentrating on lines through $\{\mathbf{0}\}$, and giving zero mass to $(\mathbf{0},\boldsymbol{\infty}]$.
- $\boldsymbol{X}$ is regularly varying on $\mathbb{E}_0=(\mathbf{0},\boldsymbol{\infty}]$ with index 2 and the limit measure gives positive mass to $(\mathbf{0},\boldsymbol{\infty}]$.

Summary:

Lesson: If the support (eg, axes) of the limit measure is less than the full space (eg, $\mathbb{E}$):

- peel away the support (axes);
- look for extreme value behavior on what's left (eg, $\mathbb{E} \setminus \{\text{axes}\} = \mathbb{E}_0$).

5.1. Directions to pursue

Antecedents: Das et al. (2011); Draisma et al. (2004); Heffernan and Resnick (2005); Ledford and Tawn (1996, 1997); Maulik and Resnick (2005); Mitra and Resnick (2011a,b); Resnick (2002)

1. Hidden regular variation (HRV)
   (a) HRV for $d = 2$.
   (b) HRV for $d > 2$. Possibly seek regular variation on a progression of decreasing of cones. Must decide how to specify sequence of cones.

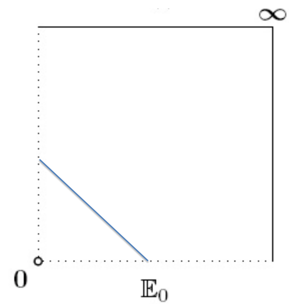
2. Hidden domain of attraction (HDA):
   - $\boldsymbol{X}$ satisfies (DOA) so that $\boldsymbol{X}^*$ satisfies (StandRegVarE).
   - (AsyIndep) holds so limit measure $\nu^*(\cdot)$ for $\boldsymbol{X}^*$ concentrates on the axes through $\mathbf{0}$.
   - However extreme value behavior other than regular variation holds in the interior of the state space. Eg, $\vee_{i=1}^d X_i^*$ has a regularly varying distribution but $\wedge_{i=1}^d X_i^*$ has a distribution in a one dimensional domain of attraction other than Fréchet.

3. What is the unit sphere? The conventional unit sphere may not be compact.

   Eg: The L_1 unit sphere is not closed.
   What plays the role of the polar coordinate transform

$$\mathbf{x} \mapsto \left(\|\mathbf{x}\|, \frac{\mathbf{x}}{\|\mathbf{x}\|} \right)$$

   and the spectral distribution.



4. More general unifying theory: Seek lower order regular variation on complement of support of the limit measure.

   - Asymptotic full dependence: limit measure concentrates on the diagonal. Remove diagonal and seek regular variation on what is left. Do we need a new theory?
   - What is the unit sphere? What takes the place of the transformation to polar coordinates?
   - What topology is appropriate? What are the *bounded* sets.

5. Estimation?

- Non-parametric approach: Does the rank transform uncover all the hidden structure?
- What sub-cones do we examine?
- How to automate in high dimensions?
- How should we infer the support of the limit measure?

6. Hidden Regular Variation

6.1. $d = 2$

Suppose $\boldsymbol{X} = (X_1, X_2)$ satisfies (DOA) and

$$U_i(x) = \frac{1}{P[X_i > x]}, \quad \boldsymbol{X}^* = \big(U_1(X_1), U_2(X_2)\big).$$

So $\boldsymbol{X}^*$ satisfies (StandRegVarE) on $\mathbb{E} = [\mathbf{0}, \boldsymbol{\infty}] \setminus \{\mathbf{0}\}$; ie,

$$nP[\frac{\boldsymbol{X}^*}{n} \in \cdot\,] \to \nu^*(\cdot).$$

$\boldsymbol{X}^*$ has *hidden regular variation* on $\mathbb{E}_0 = (0, \infty]^2$ if in addition to (StandRegVarE):

- There is a measure $\nu_0^*(\cdot)$ on $\mathbb{E}_0$; and a
- There is a sequence $b_0(n) \to \infty$ such that $b_0(n)/n \to 0$; and
- On $\mathbb{E}_0$

$$nP[\frac{\boldsymbol{X}^*}{b_0(n)} \in \cdot\,] \to \nu_0^*(\cdot). \qquad \text{(HRV E0)}$$

- Q: What sets can we insert in (HRV E0)?
- A: Sets bounded away from the deleted points; ie, sets bounded away from the axes. These are sets in a neighborhood of $\boldsymbol{\infty}$.

Consequences

- Because $b_0(n) = o(n)$, $\boldsymbol{X}^*$ and hence $\boldsymbol{X}$ must have (AsyIndep).
- For some $\alpha_0 \geq 1$,
$$b_0(n) \in RV_{1/\alpha_0}.$$
- Hence
  - Distribution tail of max is regularly varying:
$$P[X_1^* \vee X_2^* > x] \in RV_{-1},$$
    and
  - Distribution tail of min is regularly varying:
$$P[X_1^* \wedge X_2^* > x] \in RV_{-\alpha_0}.$$

This provides a strategy for detection.

Example 1: $\boldsymbol{X}^* = (X_1, X_2)$, $X_1 \perp\!\!\!\perp X_2$ and

$$P[X_i > x] = x^{-1}, \quad x > 1,\ i = 1, 2.$$

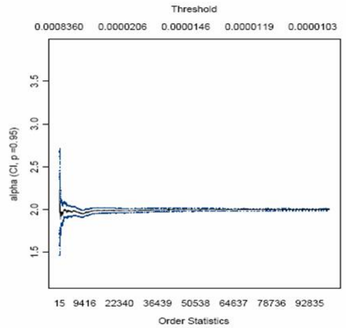
Then $\alpha_0 = 2$. Consider $\boldsymbol{X}_1, \dots, \boldsymbol{X}_{5000}$ iid.

5000 pairs of iid pareto;

$$\alpha = 1;\ \alpha_0 = 2.$$

Hill plot for minima of components.

Conclude: Maybe it is possible to detect HRV.



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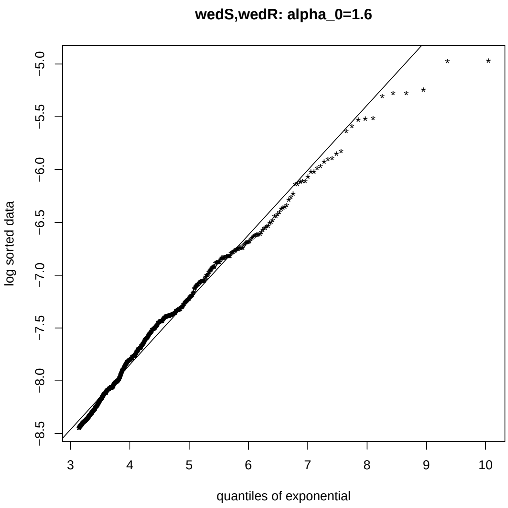
Example 2: UNC Wed (S,R): Response data where S is size of response and R is average transmission rate= size/(download time). Recall angular measure density plot seemed to indicate asymptotic independence.

- Need non-standard model.
- Standardize using rank method.
- QQ plot of minimum component of rank transformed data using 1000 upper order statistics for UNC Wed (S,R).
- Method yields $\alpha = 1$ and estimated $\hat{\alpha}_0 = 1.6$.
- Conclude: For $d = 2$, detection might be possible



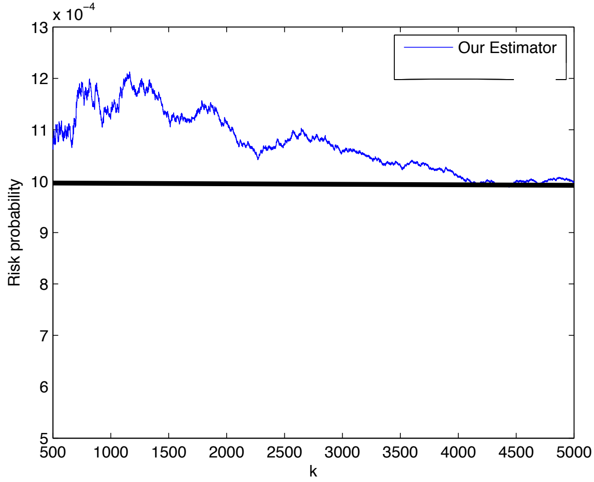
Example 3: Risk calculations.

Simulate data: $\{((X_1(n), X_2(n)); 1 \leq n \leq 5000\}$ iid where

- $X_1(n) \perp\!\!\!\perp X_2(n)$ for each n;
- $X_1(n) \sim \text{Par}(1)$, $X_2(n) \sim \text{Par}(2)$.
- Estimate the risk probability (exact value=0.001)

  $$P[X_1 > 100, X_2 > \sqrt{10}]$$

  with *spectral distribution estimator* for HRV.
- Conclude: At least in nice cases, this can work.



Hidden regular variation (continued)

6.2. $d > 2$:

One way to proceed (Mitra and Resnick, 2011b).
Recall $\mathbb{E} = [0, \infty]^d \setminus \{\mathbf{0}\}$. For $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{E}$, and $l = 1, \dots, d$, let

$$\begin{aligned} x_{(l)} &= l\text{th largest component of } x_1, \dots, x_d, \\ \mathbb{E}_{(l)} &= \{\mathbf{x} \in \mathbb{E} : x_{(l)} > 0\}. \end{aligned}$$

So

$$\begin{aligned} \mathbb{E}_{(1)} &= \{\mathbf{x} \in \mathbb{E} : \vee_{i=1}^d x_i > 0\} \\ &= \mathbb{E}, \\ \mathbb{E}_{(2)} &= \{\mathbf{x} \in \mathbb{E} : x_{(2)} > 0\} \\ &= \{\mathbf{x} \in \mathbb{E} : \text{ at least 2 components positive }\}, \end{aligned}$$

etc and

$$\mathbb{E}_{(1)} \supset \mathbb{E}_{(2)} \supset \cdots \supset \mathbb{E}_{(d)}.$$

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Def. Assume $\boldsymbol{X}^* = (X_1^*, \ldots, X_d^*)$ satisfies (StandRegVarE) with limit measure $\nu^*(\cdot)$ on $\mathbb{E}$ and scaling function $b_1(n) = n$. For $l > 1$, $\boldsymbol{X}^*$ has HRV on $\mathbb{E}_{(l)}$ if

1. For some $l > j \geq 1$ (nb. $j = 1$ is ok), $\boldsymbol{X}^*$ is regularly varying on $\mathbb{E}_{(j)}$ with scaling function $b_j(n)$ and limit measure $\nu_{(j)}(\cdot) \not\equiv 0$, on $\mathbb{E}_{(j)}$ with
$$\nu_{(j)}(\mathbb{E}_{(l)}) = 0, \qquad \nu_{(j)}(\mathbb{E}_{(l-1)}) > 0,$$
AND
2. $\boldsymbol{X}^*$ is regularly varying on $\mathbb{E}_{(l)}$ with scaling function $b_l(n)$ and limit measure $\nu_{(l)}(\cdot) \not\equiv 0$ on $\mathbb{E}_{(l)}$ and
$$b_j(n)/b_l(n) \to \infty.$$

Note $\exists\, \alpha_{(j)} \leq \alpha_{(l)}$ and
$$\nu_{(j)}(c\cdot) = c^{-\alpha_{(j)}}\nu_{(j)}(\cdot) \qquad \nu_{(l)}(c\cdot) = c^{-\alpha_{(l)}}\nu_{(l)}(\cdot),$$
so the regular variation is of lower order on the smaller cone $\mathbb{E}_{(l)}$ (which makes it hidden).

Example show:

- Could have HRV on each subcone $\mathbb{E}_{(l)}, l = 2, \dots, d$.
- For $d = 3$, could have
  - Regular variation on $\mathbb{E} = \mathbb{E}_{(1)}$ without asmptotic independence (AsyIndep).
  - No HRV on $\mathbb{E}_{(2)}$ but HRV on $\mathbb{E}_{(3)}$.
  - $\nu^*(\mathbb{E}_{(1)}) = \nu^*(\mathbb{E}_{(2)}) = \infty, \quad \nu^*(\mathbb{E}_{(3)}) = 0$.
- For $d = 3$ could have
  - $\boldsymbol{X}^*$ has regular variation on $\mathbb{E}$ with (AsyIndep).
  - HRV holds on $\mathbb{E}_{(2)}$.
  - No HRV on $\mathbb{E}_{(3)}$.
- For $d = 3$ possible that
  - $\boldsymbol{X}^*$ has regular variation on $\mathbb{E}$.
  - HRV exists on $\mathbb{E}_{(2)}$ but not on $\mathbb{E}_{(3)}$ but yet $\boldsymbol{X}^*$ has regular variation on $\mathbb{E}_{(3)}$.
  - (AsyIndep) holds but $\nu_{(2)}(\mathbb{E}_{(3)}) > 0$.

7. Hidden Domain of attraction (HDA)

(Mitra and Resnick, 2011a)
Recall: **Example 1:** Let $U \sim U(0,1)$ and define

$$\boldsymbol{X} = \Big(\frac{1}{U}, \frac{1}{1-U}\Big).$$

Properties:

- $\boldsymbol{X}$ satisfies (StandRegVarE) on $\mathbb{E} = [0,\infty]^2 \setminus \{\boldsymbol{0}\}$.
- $\boldsymbol{X}$ possesses (AsyIndep).
- $X_1 \wedge X_2 \le 2$ so $\boldsymbol{X}$ cannot have HRV.

BUT

- $X_1 \wedge X_2 \le 2$ belongs to the doa of the (reversed) Weibull EV distribution;
- A property akin to HRV holds on a cone but ...
- the cone is not a subcone of $\mathbb{E}$).

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Blood and guts:

For $\{(x_1,x_2) \in (-\infty,\infty]^2 : x_1+x_2 \le 0\}$, and large n,

$$
\begin{aligned}
nP\Big[n(X_1-2) > x_1, n(X_2-2) > x_2\Big] \\
&= nP\Big[1 - \frac{1}{2+x_2/n} < U < \frac{1}{2+x_1/n}\Big] \\
&= n\Big(\frac{1}{2+x_1/n} - \big(1 - \frac{1}{2+x_2/n}\big)\Big) \\
&= \frac{n}{2}\Big(-\frac{x_1+x_2}{2n} + O(\frac{1}{n^2})\Big) \\
&\to -(x_1+x_2)/4 \quad (n \to \infty),
\end{aligned}
$$

and if $x_1+x_2 \ge 0$,

$$nP\Big[n(X_1-2) > x_1,\ n(X_2-2) > x_2\Big] \to 0.$$

Hmmmm! Suggests concept of *hidden domain of attraction (HDA)*:

7.1. HDA: Standard case; $d = 2$.

Suppose $d = 2$ and $\boldsymbol{X} = (X_1, X_2)$ satisfies

- $X_1 \stackrel{d}{=} X_2$.
- (DOA) holds with $\boldsymbol{b}_n = (b_n, b_n) = b_n\mathbf{1}$ and $\boldsymbol{a}_n = (a_n, a_n) = a_n\mathbf{1}$ with $a_n > 0$ and
$$nP[\frac{\boldsymbol{X} - b_n\mathbf{1}}{a_n} \in \cdot] \to \nu(\cdot),$$
on $[-\infty, \infty]^2 \setminus \{-\boldsymbol{\infty}\}$ or $[0, \infty]^2 \setminus \{\mathbf{0}\}$.
- (AsyIndep) holds:
$$e^{-\nu([\boldsymbol{\infty}, \mathbf{x}]^c)} = G_1(x_1)G_2(x_2),$$
and additionally,
- there exist positive scaling and real centering constants $\{c_n\}$ and $\{d_n\}$ and a non-zero measure ν_0 on a cone $\mathbb{E}_0 = (\mathbf{0}, \boldsymbol{\infty}]$ or $(-\boldsymbol{\infty}, \boldsymbol{\infty}]$ such that
$$nP\left[(\boldsymbol{X} - d_n\mathbf{1})/c_n \in \cdot\right] \to \nu_0(\cdot) \qquad (n \to \infty). \qquad \text{(ConvE0)}$$

Then $\boldsymbol{X}$ possesses standard case HDA.

Notes:

1. It is not necessarily the case that the cone $\mathbb{E}_0 \subset \mathbb{E}$. For

$$\boldsymbol{X} = \left(\frac{1}{U}, \frac{1}{1-U}\right),$$

   we have

$$\mathbb{E} = [\mathbf{0}, \infty] \setminus \{\mathbf{0}\}, \quad \mathbb{E}_0 = \{\mathbf{x} \in (-\infty, \infty] : x_1 + x_2 \leq 0.\}.$$

2. For standard HDA, both $X_1 \vee X_2$ and $X_1 \wedge X_2$ are in one-dimensional EV doa's with EV parameters γ and γ_0. Since $X_1 \vee X_2 \geq X_1 \wedge X_2$, $\gamma \geq \gamma_0$.

3. Since $X_1 \wedge X_2$ is in a doa, set

$$U^{\wedge}(x) = \frac{1}{P[X_1 \wedge X_2 > x]}.$$

Then (ConvE0) can be expressed as *standard regular variation* on $(\mathbf{0}, \boldsymbol{\infty}]$:

$$nP\left[\left(\frac{U^{\wedge}(X_1)}{n}, \frac{U^{\wedge}(X_2)}{n}\right) \in \cdot\right] \rightarrow \tilde{\nu}_0(\cdot),$$

where $\tilde{\nu}_0(\cdot)$ is obtained from $\nu_0(\cdot)$ and satisfies

$$\tilde{\nu}^0(c\cdot) = c^{-1}\tilde{\nu}^0(\cdot), \qquad c > 0. \tag{homog}$$

(homog) allows disintegration of $\tilde{\nu}^0(\cdot)$ as a product measure in the correct coordinate system and permits definition of a spectral measure.

4. Suppose $X_1 \overset{d}{\neq} X_2$ but $\boldsymbol{X}$ satisfies (DOA). Standardize:

$$U_i(x) = \frac{1}{P[X_i > x]}, \quad i = 1, 2$$

and set

$$\boldsymbol{X}^* = (X_1^*, X_2^*) = (U_1(X_1), U_2(X_2)).$$

Since $\boldsymbol{X}$ satisfies (DOA), $\boldsymbol{X}^*$ satisfies (StandRegVarE). Assume also (AsyIndep). Can now apply the HDA definition to $\boldsymbol{X}^*$:

- Ask if there exist positive scaling and real centering constants $\{c_n\}$ and $\{d_n\}$ and a non-zero measure ν_0^* on a cone E_0 such that (ConvE0) holds with $\boldsymbol{X}^*$ replacing $\boldsymbol{X}$.
- If so, set

$$U^{*\wedge}(x) = \frac{1}{P[X_1^* \wedge X_2^* > x]},$$

  and then set

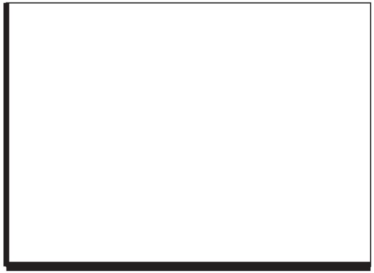
$$\boldsymbol{X}^{**} = (U^{*\wedge}(X_1^*), U^{*\wedge}(X_2^*) = \big(U^{*\wedge} \circ U_1(X_1), U^{*\wedge} \circ U_2(X_2)\big)$$

- Conclude:
  - The distribution of $\boldsymbol{X}^*$ is standard regularly varying on $[\boldsymbol{0}, \boldsymbol{\infty}] \setminus \{\boldsymbol{0}\}$ and (AsyIndep) holds.
  - $\boldsymbol{X}^{**}$ has a distribution standard regularly varying on $(\boldsymbol{0}, \boldsymbol{\infty}]$.
  - Ingenuity may be required to do estimation.

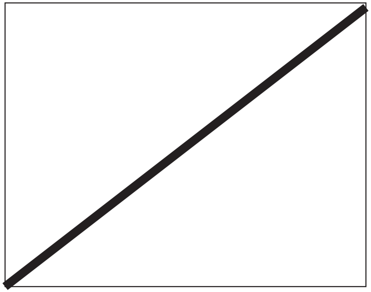
8. General approach.

(Das et al., 2011)
Compare and contrast the two situations thought to be at opposite ends of the spectrum for regularly varying distributions when $d = 2$.

1. **Asymptotic independence:** limit measure $\nu(\cdot)$ concentrates on axes through $\mathbf{0}$. Limit G is a product.



2. **Asymptotic full dependence:** limit measure $\nu(\cdot)$ concentrates on the diagonal. Limit random vector for maxima has equal components.



- In both cases, the limit measure has a *support* far smaller than $\mathbb{E} = [\mathbf{0}, \infty] \setminus \{\mathbf{0}\}$.
  - For HRV, remove support and seek a regular variation property on the complement of the support $(\mathbf{0}, \infty]$ (when $d = 2$).
  - Standard case regular variation implies limit measure $\nu^*(\cdot)$ has scaling property:

$$\nu^*(c\cdot) = c^{-1}\nu^*(\cdot), \quad c > 0,$$

    which implies

$$\text{support } \nu^* = \text{closed cone.}$$

- This suggests unifying both asymptotic independence and asymptotic full dependence and ... under one theory:
  - Identify support of the limit measure $\nu^*(\cdot)$.
  - Seek lower order regular variation on the complement of the support.

8.1. Regular variation on cones.

Abandon the one point uncompactification of the positive quadrant; exclude lines through $\boldsymbol{\infty}$. Suppose $\mathbb{F} \subset \mathbb{C} \subset [\mathbf{0}, \boldsymbol{\infty})$ are closed cones containing $\mathbf{0}$ and define

$$\mathbb{O} = \mathbb{C} \setminus \mathbb{F}.$$

→ The random vector $\boldsymbol{X} \in \mathbb{C}$ has a distribution with a regularly varying tail on $\mathbb{O}$ if $\exists\, b(t) \uparrow \infty$ and measure $\nu \not\equiv 0$ on $\mathbb{O}$ such that

$$tP[\frac{\boldsymbol{X}}{b(t)} \in \cdot\,] \to \nu(\cdot), \qquad \text{on } \mathbb{O}.$$

Let $\mathbb{F}_1$ be another closed cone containing $\mathbf{0}$ and set

$$\mathbb{O}_1 = \mathbb{C} \setminus (\mathbb{F} \cup \mathbb{F}_1).$$

→ The random vector $\boldsymbol{X} \in \mathbb{C}$ has a distribution with hidden regular variation on $\mathbb{O}_1$ if there is regular variation on $\mathbb{O}$ AND if $\exists\, b_1(t) \uparrow \infty$ and a measure $\nu_1(\cdot) \not\equiv 0$ on $\mathbb{O}_1$ such that

$$tP[\frac{\boldsymbol{X}}{b_1(t)} \in \cdot\,] \to \nu_1(\cdot), \qquad \text{on } \mathbb{O}_1,$$

AND

$$b(t)/b_1(t) \to \infty$$

(which makes the behavior on $\mathbb{O}_1$ hidden).

Examples for $d = 2$:

1. Regular variation on the positive quadrant with(AsyIndep):

$$\mathbb{C} = [\mathbf{0}, \infty), \quad \mathbb{F} = \{\mathbf{0}\}, \quad \mathbb{O} = \mathbb{C} \setminus \mathbb{F} = [\mathbf{0}, \infty) \setminus \{\mathbf{0}\}.$$

   HRV on $(\mathbf{0}, \infty)$:

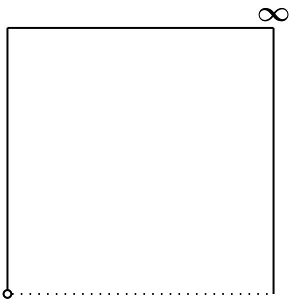
$$\begin{aligned} \mathbb{F}_1 =& \{(x, 0) : x > 0\} \cup \{(0, x) : x > 0\}, \\ \mathbb{O}_1 =& \mathbb{C} \setminus (\mathbb{F} \cup \mathbb{F}_1) = [\mathbf{0}, \infty) \setminus \{\mathbf{0}\} \cup \{\text{lines through } \mathbf{0}\} \\ =& (\mathbf{0}, \infty). \end{aligned}$$

2. Regular variation on the positive quadrant with conditional extreme value (CEV) model:

$$\mathbb{C} = [\mathbf{0}, \infty), \ \mathbb{F} = \{\mathbf{0}\},$$
$$\mathbb{O} = \mathbb{C} \setminus \mathbb{F} = [\mathbf{0}, \infty) \setminus \{\mathbf{0}\}.$$

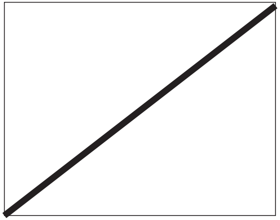
CEV on $\mathbb{D}_{\sqcap}$:

$$\begin{aligned} \mathbb{F}_1 =& \{(x, 0) : x > 0\}, \\ \mathbb{O}_1 =& \mathbb{C} \setminus (\mathbb{F} \cup \mathbb{F}_1) \\ =& [\mathbf{0}, \infty) \setminus \big(\{\mathbf{0}\} \cup \{(x, 0) : x > 0\}\big) \\ =& [0, \infty) \times (0, \infty) \\ =:& \mathbb{D}_{\sqcap}. \end{aligned}$$



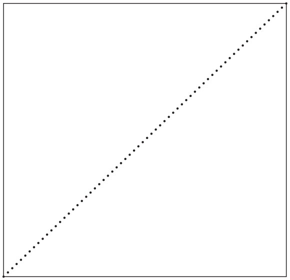
3. Asymptotic full dependence:

Regular variation on $[\mathbf{0},\boldsymbol{\infty})\setminus\{\mathbf{0}\}$ with limit measure concentrating on diagonal.



$$\begin{aligned}\mathbb{C} =&[\mathbf{0},\infty),\ \mathbb{F} = \{\mathbf{0}\},\\ \mathbb{O} =&\mathbb{C}\setminus\mathbb{F} = [\mathbf{0},\boldsymbol{\infty})\setminus\{\mathbf{0}\}.\end{aligned}$$

Remove diagonal:



$$\begin{aligned}\mathbb{F}_1 =&\{(x,x): x>0\},\\ \mathbb{O}_1 =&\mathbb{C}\setminus(\mathbb{F}\cup\mathbb{F}_1)\\ =&[\mathbf{0},\boldsymbol{\infty})\setminus\{(x,x): x\geq 0\}\end{aligned}$$

Example 3 (continued): Asymptotic full dependence.

Suppose $\boldsymbol{X} = (X_1, X_2)$ is regularly varying on $[\mathbf{0}, \boldsymbol{\infty}) \setminus \{\mathbf{0}\}$ with asymptotic full dependence so the limit measure $\nu(\cdot$ concentrates on $\{(x,x) : x > 0\}$. Suppose

$$X_i = \text{ one period loss of financial instrument } I_i.$$

Construct the portfolio:

- Buy one unit of I_1. (Go long.)
- Sell one unit of I_2. (Go short.)

One period loss for the portfolio is

$$L = X_1 - X_2$$

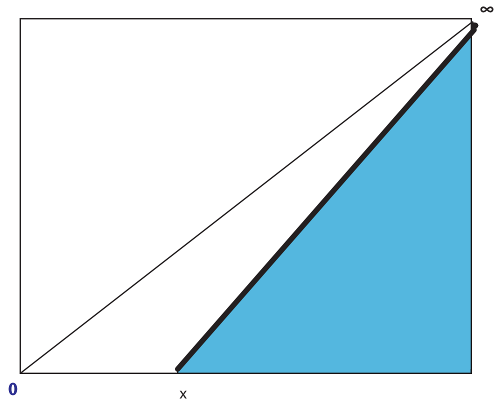
and for large x, seek

$$P[X_1 - X_2 > x].$$

Under asymptotic full dependence, limit measure concentrates on the line $\{(x,x) : x > 0\}$ so we estimate probability as 0:

$$\widehat{P}[X_1 - X_2 > x] = 0.$$

Conclude: A more general theory has applicability.



8.2. Consequences:

- Need for a more general theory of HRV which covers
  - Asymptotic independence.
  - Asymptotic full dependence.
  - Other cases where the support of limit measure is strictly smaller than the state space.
- If doing HRV estimation of the probability of the blue region under asymptotic full dependence:
  - Remove diagonal.
  - But: then the blue region is not relatively compact if include lines through ∞ in state space.
  - $\Rightarrow$ need to remove lines through ∞ and use different topology [Hult and Lindskog (2006)].
  - Demise of the one-point uncompactification?

8.3. Polar coordinate transform and unit spheres:

- A limit measure ν for standard regular variation on cone $\mathbb{O}$ satisfies the scaling property:

$$\nu(c\cdot) = c^{-1}\nu(\cdot).$$

- If $T : \mathbb{O} \mapsto (0,\infty) \times \mathbb{A}$ is a bijection $\mathbf{x} \mapsto (r, \boldsymbol{a})$ satisfying

$$T(c\mathbf{x}) = (cr, \boldsymbol{a}),$$

then the scaling property of $\nu(\cdot)$ translates to a disintegration property in the new coordinates:

$$\nu \circ T^{-1}(dr, d\boldsymbol{a}) = r^{-2} dr\, S(d\boldsymbol{a})$$

is a product measure on $(0,\infty) \times \mathbb{A}$ and

$$S(\cdot) := \nu \circ T^{-1}\big((1,\infty) \times (\cdot)\big)$$

is the *spectral measure*. Would like $T^{-1}\big((1,\infty) \times (\cdot)\big)$ to be relatively compact and then S is a finite (probability) measure.

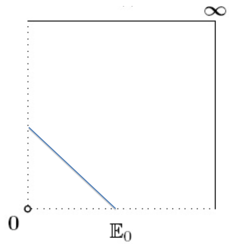
Examples:

– $[\mathbf{0},\boldsymbol{\infty}) \setminus \{\mathbf{0}\}$: Set $T(\mathbf{x}) = (\|\mathbf{x}\|, \mathbf{x}/\|\mathbf{x}\|)$ and $\mathbb{A}$ is the (conventional) unit sphere which is compact:

$$\mathbb{A} = \{\mathbf{x} \in [\mathbf{0},\boldsymbol{\infty}) : \|\mathbf{x}\| = 1\}.$$

– $(\mathbf{0},\infty)$:

The transformation in the previous item does not lead to a relatively compact unit sphere.



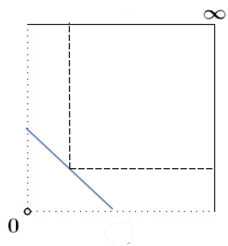
so use

$$T(\mathbf{x}) = (x_1 \wedge x_2, \frac{\mathbf{x}}{x_1 \wedge x_2}).$$

The new unit sphere

$$\mathbb{A} = \{\mathbf{x} \in (\mathbf{0},\infty) : x_1 \wedge x_2 = 1\},$$

being bounded away from $\mathbf{0}$ is relatively compact and so is



$$T^{-1}((1,\infty)\times(\cdot)) = \{\mathbf{x} \in (\mathbf{0},\infty) : x_1 \wedge x_2 \geq 1\}.$$

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– $\mathbb{O} = \mathbb{C} \setminus \mathbb{F}$ for closed cones $\mathbb{C}$, $\mathbb{F}$. Set

$$T(\mathbf{x}) = \Big(\text{dist}(\mathbf{x}, \mathbb{F}), \frac{\mathbf{x}}{\text{dist}(\mathbf{x}, \mathbb{F})}\Big), \quad \mathbf{x} \in \mathbb{O}.$$

Then

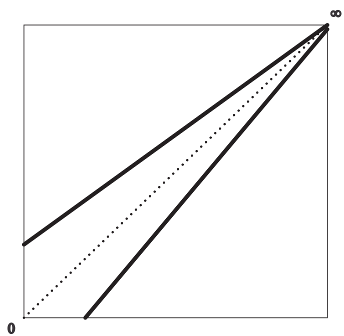
$$\{\mathbf{x} \in \mathbb{O} : \text{dist}(\mathbf{x}, \mathbb{F}) \geq 1\}$$

is bounded away from the deleted $\mathbb{F}$ and is relatively compact and

$$\mathbb{A} := \{\mathbf{x} \in \mathbb{O} : \text{dist}(\mathbf{x}, \mathbb{F}) = 1\}$$

serves as a unit sphere.

Example: $\mathbb{F}$ is the diagonal. Unit sphere is parallel lines to the diagonal.



9. Final remarks.

- Practical?
  - Limitations of asymptotic methods: rates of convergence?
  - Instead of estimating a risk probability as 0, estimate is a very small number.
- Need for more formal inference for estimation including confidence statements.
- General HRV technique requires knowing the support of the limit measure. Estimate support?
- High dimension problems? How to sift through different possible subcones?
- How to go from standard to more realistic non-standard case; still some inference problems.

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