Notes

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1 Black-Scholes Formula

In this section we apply the theory of Stochastic Calculus to prove the well known *Black-Scholes formula* about the option pricing of a stock asset whose price dynamics are modeled as a geometric Brownian Motion.

1.1 Financial Arbitrage

In finance one main assumption about the market is that the prices are determined by the principle of "*no arbitrage opportunity*".

Definition 1. *An* ***arbitrage opportunity*** *is any circumstance where the simultaneous purchase and sale of the related security is guaranteed to produce a risk-less profit.*

According to this principle it follows that any two investments with identical payout streams must have the same price.

If this were not true, one could take advantage of an arbitrage opportunity by buying the security at the lower price and simultaneously selling it at the higher price. Being th payout streams equal, s/he could compensate the two in-out payout flows such as to gain in a risk-less way an amount of money equal to the difference of the two prices. By repeating this for a very large number of units of the same security s/he could become with no risk infinitely rich.

Therefore the principle of "*no arbitrage opportunity*" gives a practical way to determine the price of a new financial instrument, such as for example an option. To see this assume we were able to replicate the new investment by another one with known price and identical payout stream, then the financial instrument we want to value should have necessarily the same price.

Before analyzing the Black-Scholes model we give two examples of replication that yield to pricing two simple financial instruments: the "*Forward Contracts*" and the "*Binomial Option*".

In what follows we are going to assume that the market is ideal, i.e. there are no transaction costs and the interest rates are symmetric that means we can borrow and lend money at the same interest rates.

1.2 Forward Contract

Definition 2. *A* ***forward contract*** *is an agreement to buy a commodity at a given future time T at a fixed price K.*

A Forward Contract obviously has a price, let us say F, and in the following we are going to make a replica of this investment in order to determine the value of F.

The replicating portfolio consists in buying at time 0 one unit of the commodity, whose price is $S(0) = S$, and selling $e^{-r\,T}K$ of bond (that is equivalent to borrowing a same amount of money). Calling $V(0)$ the value of the portfolio at time 0 (i.e. the amount of money we would get if we decided to sell it completely), and $V(T)$ its value at time T we get the following comparison table

	Time 0	Time T
Forward Contract	F	$S(T) - K$
Replicating Portfolio	$S - e^{-r\,T}K$	$S(T) - K$

Notice that the value of the forward contract at time T is $S(T) - K$, because we are required to buy at time T the commodity at price K for a net gain of $S(T) - K$. Equating the pay-out flows we immediately get that $F = S - e^{-r\,T}K$.

1.3 Binomial Option

In this section we try to price an option assuming a simplified scenario where there exist only time 0 and time T and two possible values for the future price of the stock at time T, say $S(T)$. The commodity, that to fix ideas we suppose to be a stock, has price $S(0)$ at time 0 and at time T it may have price S_U with probability p or price S_D with probability $1-p$. The option, that costs an unknown quantity O, gives the right but not the obligation to buy the stock at time T at a price K with $S_D < K < S_U$ so that owning it has a worth of $(S(T)-K)^+$ with $(a)^+ = \max\{0, a\}$.

The replicating portfolio again consists in buying at time 0 some units of the "*underlying*" (the stock), at price $S(0) = S$, and some units of bond at price $\beta(0) = \beta$. Negative quantities of bond actually means that we sell units of bond that is the same as borrowing money. Assuming that the quantities of stock and bond are unknown and equal to a and b we have that the values of our portfolio in the present and the future are given by

$$V(0) = aS(0) + b\beta(0) = aS(0) + b\beta \quad \text{and} \quad V(T) = aS(T) + b\beta(T) = aS(T) + b\,\beta\,e^{r\,T}.$$

Comparing with the values of owning the option we get the following resuming table

	Time 0	Time T	
		UP - w.p. p	DOWN - w.p. $1-p$
Binomial Option	O	$S_U - K$	0
Replicating Portfolio	$a\,S + b\,\beta$	$a\,S_U + b\,\beta\,e^{r\,T}$	$a\,S_D + b\,\beta\,e^{r\,T}$

Equating the pay-out streams we get

$$a = \frac{S_U - K}{S_U - S_D} \quad \text{and} \quad b = -\frac{S_U - K}{S_U - S_D}\frac{S_D}{\beta}e^{-r\,T}$$

and therefore the binomial option has to be priced

$$O = \frac{S_U - K}{S_U - S_D}[S - S_D\,e^{-r\,T}].$$

1.4 Black-Scholes Model

The Black-Scholes model is a generalization of the binomial model, that assumes that the changes in the stock price happen in continuous time. The model chosen for the price dynamics of the underlying is represented by the following stochastic differential equation

$$dS(t) = \mu\,S(t)\,dt + \sigma\,S(t)\,dB(t) \tag{1}$$

where we assume that the price at time 0 is $S(0) = S$ and where $B(t)$ is the standard Brownian motion. As for the bond, its dynamics are described by a deterministic ordinary differential equation, i.e.

$$d\beta(t) = r\,\beta(t)\,dt,$$

whose solution is simply $\beta(t) = \beta\,e^{r\,t}$ where we assumed $\beta(0) = \beta$.

The specific option we are considering has the property to give us the right but not the obligation, to *buy* the underlying at fixed time T, called the *execution time*, at price K, called *strike price*. This financial instrument is generally know as *European Call Option*.

In the same way as we did for the binomial model we assume that the replicating portfolio consists in holding at time t a quantity $a(t)$ of stock and a corresponding quantity $b(t)$ of bonds. The value of the option at time 0 is the unknown price O while its value at time T is $(S(T)-K)^+$.

Since the payout stream of the option investment allows payment only at the beginning, O, and at the end $(S(T)-K)^+$, the replicating portfolio cannot allow in-out flow of money during the interval $(0,T)$. This means that if we decide at time t to sell an amount of stock we hold, say $da(t)$, the amount of money we would receive, $S(t)\,da(t)$ has to be fully invested in buying units of bonds $db(t)$ for a total cost $-\beta(t)\,db(t)$, such that

$$S(t)\,da(t) = -\beta(t)\,db(t). \tag{2}$$

Calling $V(t)$ the value of the portfolio at time t we have that it is given by

$$V(t) = a(t)\,S(t) + b(t)\,\beta(t). \tag{3}$$

By differentiating (3) we get

$$dV(t) = a(t)\,dS(t) + S(t)\,da(t) + \beta(t)\,db(t) + b(t)\,d\beta(t) \tag{4}$$

and using (2) we get the so called "*self-financing condition*"

$$dV(t) = a(t)\,dS(t) + b(t)\,d\beta(t) \tag{5}$$

that simply means that the changes in the value of the portfolio are not due to the conversions of stock units in bond units but only to the changes of their market prices.

For the Black-Scholes model the following table shows the comparison between the value of holding an European Call Option and the value of the replicating portfolio

	Time 0	Time T	
European Call Option	O	$(S(T) - K)^+$	
Replicating Portfolio	$V(0)$	$V(T)$	

and we see that our problem reduces to determine the value of $V(0)$ with the condition that $V(T) = (S(T) - K)^+$.

To find a solution $V(t)$ to the SDE (5) we look for solution of kind $V(t) = f(t, S(t))$ for a given twice differentiable function $f(t, x)$.

Substituting this in (5) and using Itô formula we get

$$dV(t) = df(t, S(t)) = f_t(t, S(t))\,dt + f_x(t, S(t))\,dS(t) + \frac{1}{2} f_{xx}(t, S(t))\,(dS(t))^2.$$

Using (1) we get that $(dS(t))^2 = \sigma^2\,S^2(t)\,dt$ and therefore after collecting terms we get

$$dV(t) = df(t, S(t)) = [f_t(t, S(t)) + \mu\,f_x(t, S(t))\,S(t) + \frac{1}{2} f_{xx}(t, S(t))\,\sigma^2\,S^2(t)]\,dt + \sigma\,f_x(t, S(t))\,S(t)\,dB(t). \tag{6}$$

Substituting the expression for $dS(t)$ and $d\beta(t)$ in equation (5) we similarly get

$$dV(t) = [a(t)\,\mu\,S(t) + b(t)\,r\,\beta(t)]\,dt + a(t)\,\sigma\,S(t)\,dB(t) \tag{7}$$

and matching its coefficients with the ones of equation (6) we get

$$a(t) = f_x(t, S(t)) \tag{8}$$

$$b(t) = \frac{1}{r\,\beta(t)} \left[f_t(t, S(t)) + \frac{1}{2} f_{xx}(t, S(t))\,\sigma^2\,S^2(t) \right] \tag{9}$$

Finally by substituting the above expression for $a(t)$ and $b(t)$ in equation (3) we get

$$f(t, S(t)) = S(t)\,f_x(t, S(t)) + \frac{1}{r} \left[f_t(t, S(t)) + \frac{1}{2} f_{xx}(t, S(t))\,\sigma^2\,S^2(t) \right]$$

with

$$f(T, S(T)) = (S(T) - K)^+$$

and therefore the function we are looking for is the solution of the PDE

$$f_t(t, x) = -\frac{1}{2}\,\sigma^2\,x^2\,f_{xx}(t, x) - r\,x\,f_x(t, x) + r\,f(t, x) \tag{10}$$

with terminal boundary condition

$$f(T, x) = k(x), \tag{11}$$

with $k(x) = (x - K)^+$.

1.5 Solving the PDE

In this section we try to find the solution of the equation (10) that we shorty rewrite in the following way

$$f_t = -\frac{1}{2}\,\sigma^2\,x^2\,f_{xx} - r\,x\,f_x + r\,f, \tag{12}$$

and with boundary condition (11).

First we change the terminal boundary condition into an initial condition by changing the time variable and using $\tau(t) = T - t$. We change also the space variable x by using the transformation $y(x) = \ln(x)$ that allows to get rid of the factor terms x and x^2. Define $g(\tau, y) = f(T - \tau, e^y)$, it follows that $f(t, x) = g(T - t, \ln(x))$ such that the following relations hold true

$$\begin{aligned}
f_t(t,x) &= -g_\tau(\tau(t), y(x)) && \text{and in short form} \quad f_t = -g_\tau \\
f_x(t,x) &= \frac{1}{x} g_y(\tau(t), y(x)) && f_x = \frac{1}{x} g_y \\
f_{xx}(t,x) &= \frac{1}{x^2} g_{yy}(\tau(t), y(x)) - \frac{1}{x^2} g_y(\tau(t), y(x)) && f_{xx} = \frac{1}{x^2} g_{yy} - \frac{1}{x^2} g_y
\end{aligned}$$

In terms of $g(\tau, y)$ the differential equation (12) becomes

$$g_\tau = \frac{1}{2}\,\sigma^2\,g_{yy} + \left(r - \frac{1}{2}\,\sigma^2\right) g_y - r\,g, \tag{13}$$

and with initial boundary condition

$$g(0, y) = k(e^y) = (e^y - K)^+. \tag{14}$$

By letting $h(\tau, y) = e^{\alpha\,\tau + \gamma\,y} g(\tau, y)$ we have that

$$\begin{aligned}
h_\tau &= \alpha\,h + e^{\alpha\,\tau+\gamma\,y} g_\tau \\
h_y &= \gamma\,h + e^{\alpha\,\tau+\gamma\,y} g_y \\
h_{yy} &= 2\,\gamma\,h_y - \gamma^2\,h + e^{\alpha\,\tau+\gamma\,y} g_{yy}
\end{aligned}$$

yielding to

$$\begin{aligned}
e^{\alpha\,\tau+\gamma\,y} g_\tau &= h_\tau - \alpha\,h \\
e^{\alpha\,\tau+\gamma\,y} g_y &= h_y - \gamma\,h \\
e^{\alpha\,\tau+\gamma\,y} g_{yy} &= h_{yy} - 2\,\gamma\,h_y - \gamma^2\,h.
\end{aligned}$$

Multiplying equation (13) on both sides by $e^{\alpha\,\tau+\gamma\,y}$ and using last expressions we finally get

$$h_\tau = \frac{1}{2}\,\sigma^2\,h_{yy} + \left[r - \frac{1}{2}\,\sigma^2 - \gamma\,\sigma^2\right] h_y + \left[\alpha + \frac{1}{2}\,\gamma^2\,\sigma^2 - \gamma\left(r - \frac{1}{2}\,\sigma^2\right) - r\right] h$$

with initial boundary condition

$$h(0, y) = e^{\gamma\,y}\,k(e^y) = e^{\gamma\,y}\,(e^y - K)^+.$$

Making the special choice of

$$\begin{aligned}
\gamma^* &= \frac{1}{\sigma^2}\left(r - \frac{1}{2}\sigma^2\right) \\
\alpha^* &= r + \frac{1}{2\sigma^2}\left(r - \frac{1}{2}\sigma^2\right)^2
\end{aligned}$$

for the constants α and γ we finally obtain

$$h_\tau = \frac{1}{2}\,\sigma^2\,h_{yy} \tag{15}$$

with initial boundary condition

$$h(0,y) = e^{\gamma^* y}\, k(e^y) = e^{\gamma^* y}\, (e^y - K)^+. \tag{16}$$

It is known [see the notes on the Backword Diffusion Equations about the (μ, σ)-BM, $W(t)$] that the solution of the *heat equation* (15) is given by

$$h(\tau, y) = \frac{1}{\sqrt{2\,\pi\,\sigma^2\,\tau}} \int_{-\infty}^{+\infty} h(0,x)\, e^{-\frac{(y-x)^2}{2\sigma^2\tau}}\, dx = \int_{-\infty}^{+\infty} h(0, y - z\sigma\sqrt{\tau})\, d\Phi(z) \tag{17}$$

where $\Phi(z) = \mathbb{P}\{Z \leq z\}$ with $Z \sim \mathrm{N}(0,1)$ a standard Normal random variable.

Substituting back $f(t,x) = e^{-\alpha^*\,(T-t)}\, x^{-\gamma^*}\, h(T-t, \ln(x))$, after some algebraic manipulations we get

$$f(t,x) = x\,\Phi\left(\frac{\ln(x/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) - K\,e^{-r(T-t)}\,\Phi\left(\frac{\ln(x/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) \tag{18}$$

and therefore the price of the option will be

$$O = f(0,S) = S\,\Phi\left(\frac{\ln(S/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) - K\,e^{-r(T-t)}\,\Phi\left(\frac{\ln(S/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right),$$

with $S = S(0)$.

In addition thanks to the derivation did before we obtain as a side product also the way how to replicate the option by using a portfolio made of the stock and the bond. Indeed the amount of stock units held at time t has to be equal to

$$a(t) = f_x(t, S(t)) = \Phi\left(\frac{\ln(S(t)/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)$$

and the units of bonds

$$b(t) = -\frac{K}{\beta}\, e^{-r\,T}\,\Phi\left(\frac{\ln(S(t)/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right).$$