Notes

Abril 13^{th}, 2010

1 Stochastic Calculus

As we have seen in previous lessons, the stochastic integral with respect to the Brownian motion shows a behavior different from the classical Riemann-Stieltjes integral, and this difference pops up thanks to the non-null limit of the following Riemann sum

$$\sum_{k=1}^{|\Pi|}[B(t_k) - B(t_{k-1})]^2 \xrightarrow[||\Pi||\downarrow 0]{\mathcal{L}^2} t, \tag{1}$$

In the following section we prove that the $\mathcal{L}^2$ convergence in (1) holds and that its limit is given by the quadratic variation of the Brownian motion over the interval $[0, t]$

1.1 Total Variation and Quadratic Variation

Definition 1. *Given a function $f(t)$, $t \geq 0$, the total variation of f over the interval $[0, t]$, $V_t(f)$, is defined as*

$$V_t(f) = \sup_{\Pi\in\mathcal{P}} \sum_{k=1}^{|\Pi|} |f(t_k) - f(t_{k-1})|. \tag{2}$$

If $V_t(f) < \infty$ for any $t \geq 0$ we say that f is of Bounded Variation and we denote it by writing $f \in BV$.

Definition 2. *Given a function $f(t)$, $t \geq 0$, the quadratic variation of f over the interval $[0, t]$, $Q_t(f)$, is defined as*

$$Q_t(f) = \sup_{\Pi\in\mathcal{P}} \sum_{k=1}^{|\Pi|} |f(t_k) - f(t_{k-1})|^2. \tag{3}$$

Assuming that f is a continuous function, by noticing that for any $\Pi \in \mathcal{P}$

$$\sum_{k=1}^{|\Pi|} |f(t_k) - f(t_{k-1})| \geq \frac{\sum_{k=1}^{|\Pi|} |f(t_k) - f(t_{k-1})|^2}{\max_{1\leq k\leq|\Pi|} |f(t_k) - f(t_{k-1})|}$$

we have that if $f \in BV$ then $Q_t(f) \equiv 0$ and that any function with non-zero quadratic variation has infinite total variation.

Remark 1. *In case of a stochastic process $\{X(t), t \geq 0\}$, the definitions for total variation and quadratic variation stay the same with the only remark that the limits are intended in probability sense.*

Proposition 1. *The quadratic variation function of the standard Brownian motion, $B(t)$, is given by*

$$Q_t(B) = t.$$

Proof. Given that $\mathbb{E}[[B(t_k) - B(t_{k-1})]^2] = \mathbb{V}\text{ar}[B(t_k) - B(t_{k-1})] = (t_k - t_{k-1})$ we immediately get

$$\mathbb{E}\left[\sum_{k=1}^{|\Pi|}[B(t_k) - B(t_{k-1})]^2\right] = t.$$

It is enough to prove that

$$\mathbb{V}\text{ar}\left[\sum_{k=1}^{|\Pi|}[B(t_k) - B(t_{k-1})]^2\right] \to 0 \quad \text{as } ||\Pi|| \to 0$$

to get that the convergence in (1) holds true.

We have

$$\mathbb{V}\text{ar}\left[\sum_{k=1}^{|\Pi|}[B(t_k)-B(t_{k-1})]^2\right]=\sum_{k=1}^{|\Pi|}\mathbb{V}\text{ar}\left[[B(t_k)-B(t_{k-1})]^2\right]=2\sum_{k=1}^{|\Pi|}[t_k-t_{k-1}]^2$$

Therefore

$$\mathbb{V}\text{ar}\left[\sum_{k=1}^{|\Pi|}[B(t_k)-B(t_{k-1})]^2\right]\to Q_t(t)=0,$$

where in the last equality we used the fact that the linear function t has finite total variation $V_t(t)=t$ and therefore for the remark above zero quadratic variation.

Noticing that the $\mathcal{L}^2$-convergence implies the convergence in probability, we get the result. □

1.2 The Itô integral - Properties

Definition 3. *Given the standard Brownian motion $\{B(t),t\geq 0\}$ and an adapted stochastic process $\{X(t),t\geq 0\}$ satisfying the condition*

$$\int_0^t \mathbb{E}[X^2(s)]ds<\infty$$

the Itô integral, $I_t(X)$ is defined as

$$I_t(X)=\int_0^t X(s)\,dB(s)=\lim_{||\Pi||\to 0}\sum_{k=1}^{|\Pi|}X(t_{k-1})[B(t_k)-B(t_{k-1})]$$

where the limit is meant in $\mathcal{L}^2$ sense.

Proposition 2. *The Itô integral shares the following properties*

a) $\mathbb{E}\left[\int_0^t X(s)\,dB(s)\right]=0$

b) $\mathbb{V}\text{ar}\left[\int_0^t X(s)\,dB(s)\right]=\int_0^t \mathbb{E}[X^2(s)]\,ds$ ***(Itô isometry)***

c) $\int_0^t a_1\,X_1(s)+a_2X_2(s)\,dB(s)=a_1\int_0^t X_1(s)\,dB(s)+a_2\int_0^t X_2(s)\,dB(s)$ ***(Linearity)***

d) $M(t)=M(0)+\int_0^t X(s)\,dB(s)$ *is a continuous Martingale* ***(Martingale property)***

In addition it is easy to prove that

$$\int_0^t a\,dB(s)=a\,B(t) \tag{4}$$

and

$$\int_0^t B(s)\,dB(s)=\frac{B^2(t)}{2}-\frac{t}{2}. \tag{5}$$

Definition 4. *An Itô process, $\{X(t),0\leq t\leq T\}$, is any stochastic process that may be written in the following form*

$$X(t)=X(0)+\int_0^t g(s)\,ds+\int_0^t h(s)\,dB(s) \tag{6}$$

where $g(\omega,s)$ and $h(\omega,s)$ are two adapted stochastic processes such that

$$\mathbb{P}\{\int_0^T |g(s)|\,ds<\infty\}=1$$
$$\mathbb{P}\{\int_0^T |h(s)|^2\,ds<\infty\}=1.$$

Equation (6) can be written in differential form in the following way

$$dX(t) = g(t)\,dt + h(t)\,dB(t) \tag{7}$$

The integration of a deterministic function with respect to the Brownian motion yields to a Gaussian process whose parameter functions are easy to compute, as it is shown in the following proposition.

Proposition 3. *Given a deterministic function $f(t)$ the Itô process*

$$I_t(f) = \int_0^t f(u)\,dB(u)$$

is a Gaussian process with zero mean function and covariance function

$$\mathbb{C}\mathrm{ov}(I_t(f), I_s(f)) = \mathbb{E}\left[\int_0^t \int_0^s f(\tau)\,f(\sigma)\,dB(\sigma)dB(\tau)\right] = \int_0^{t\wedge s} f^2(u)du.$$

Proof. Use the Itô isometry and the independence of the increments of the Brownian motion. □

An important formula to compute the values of the stochastic integrals is the *Itô formula* given in the following proposition.

Proposition 4 (Itô Lemma)**.** *Given an Itô process $\{X(t), t \geq 0\}$ and a function $f(t,x) \in \mathcal{C}^2(\mathbb{R}^+, \mathbb{R})$ then the following relation holds true*

$$f(t, X(t)) = f(0, X(0)) + \int_0^t f_t(s, X(s))\,ds + \int_0^t f_x(s, X(s))\,dX(s) + \frac{1}{2}\int_0^t f_{xx}(s, X(s))\,(dX(s))^2, \tag{8}$$

where $f_t(t,x) = \partial_t f(t,x)$, $f_x(t,x) = \partial_x f(t,x)$ and $f_{xx}(t,x) = \partial_x^2 f(t,x)$.

In the formula above $dX(s)$ is given by equation (7) while $(dX(s))^2 = h^2(s)\,ds$ with $h(t)$ being the function appearing in the definition of $X(t)$ in (6). $(dX(s))^2$ can be formally obtained by computing $dX(s) \times dX(s)$ from equation(7) and using the following resuming table

$\times$	dt	$dB(t)$
dt	0	0
$dB(t)$	0	dt

In differential form equation (8) is written in the following way

$$df(t, X(t)) = f_t(t, X(t))\,dt + f_x(t, X(t))\,dX(t) + \frac{1}{2}f_{xx}(t, X(t))\,(dX(t))^2. \tag{9}$$

Remark 2. *For the case $X(t)$ is the standard Brownian motion, the Itô formula (8) simplifies in*

$$f(t, B(t)) = f(0,0) + \int_0^t [f_t(s, B(s)) + \frac{1}{2}f_{xx}(s, B(s))]\,ds + \int_0^t f_x(s, B(s))\,dB(s) \tag{10}$$

and in differential form

$$df(t, B(t)) = [f_t(t, B(t)) + \frac{1}{2}f_{xx}(t, B(t))]\,dt + f_x(t, B(t))\,dB(t). \tag{11}$$

Example 1. *Using Itô formula it is easy to compute the value of $\int_0^t B(s)\,dB(s)$. Choosing $f(t,x) = x^2$, such that $f_t(t,x) = 0$, $f_x(t,x) = 2x$ and $f_{xx}(t,x) = 2$, by (10) we get*

$$B^2(t) = \int_0^t ds + \int_0^t 2\,B(s)\,dB(s)$$

and therefore

$$\int_0^t B(s)\,dB(s) = \frac{B^2(t)}{2} - \frac{t}{2}$$

that agrees with (5).

1.3 Chain Rule

The following propositions underline the differences between the stochastic integral and the classical Riemann-Stiltjes integral.

Proposition 5 (Riemann-Stiltjes' Chain Rule)**.** *Given a continuous differentiable function $f \in \mathcal{C}^1(\mathbb{R})$ and a BF function $\{x(t), t \geq 0\}$, the following relation holds true*

$$f(x(t)) = f(x(0)) + \int_0^t f'(x(s))\,dx(s).$$

Proposition 6 (Itô's Chain Rule)**.** *Given a continuous twice differentiable function $f \in \mathcal{C}^2(\mathbb{R})$ and $\{X(t), t \geq 0\}$ a function with finite quadratic variation, the following relation holds true*

$$f(X(t)) = f(X(0)) + \int_0^t f'(X(s))\,dX(s) + \frac{1}{2}\int_0^t f''(X(s))\,(dX(s))^2$$

2 Application of the Stochastic Calculus

2.1 The Geometric Brownian Motion

In this section we look for the solution of the following SDE

$$\frac{dX(t)}{X(t)} = \mu\,dt + \sigma\,dB(t) \tag{12}$$

that can also be rewritten as

$$dX(t) = \mu\,X(t)\,dt + \sigma\,X(t)\,dB(t). \tag{13}$$

We use as test function $X(t) = f(t, B(t))$ and applying the Itô formula (9) we get

$$df(t, B(t)) = [f_t(t, B(t)) + \frac{1}{2} f_{xx}(t, B(t))]\,dt + f_x(t, B(t))\,dB(t). \tag{14}$$

Matching the coefficients of dt and $dB(t)$ we obtain

$$\mu f(x,t) = f_t(t,x) + \frac{1}{2} f_{xx}(t,x) \tag{15}$$

$$\sigma f(x,t) = f_x(t,x) \tag{16}$$

$$\tag{17}$$

and taking the derivative of (16) and substituting it in (15) we get

$$f_t(t,x) = \left[\mu - \frac{\sigma^2}{2}\right] f(t,x)$$

that solved gives

$$f(t,x) = f(0,x)\,e^{\left(\mu - \frac{\sigma^2}{2}\right)t}.$$

Substituting last expression in (16) we get

$$f_x(0,x) = \sigma f(0,x)$$

that has solution

$$f(0,x) = f(0,0)e^{\sigma\,x}.$$

Finally the solution of (13) is given by

$$X(t) = f(t, B(t)) = X(0)\,e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\,B(t)}.$$

Alternative derivation.
Another way to get a solution of (16) is by computing the correct differential of the function $\ln(X(t)$ that differs from $dX(t)/X(t)$ of the classical calculus.

Using Itô formula (9) we have that

$$d\ln(X(t)) = \frac{dX(t)}{X(t)} - \frac{(dX(t))^2}{2X^2(t)},$$

and substituting the expressions of $dX(t)$ and $(dX(t))^2$ obtained from (13)

$$\begin{aligned} dX(t) &= \mu\, X(t)\, dt + \sigma\, X(t)\, dB(t) \\ (dX(t))^2 &= \mu^2\, X^2(t)\, (dt)^2 + \sigma^2\, X^2(t)\, (dB(t))^2 + 2\mu\sigma X^2(t)\, dB(t) \times dt \\ &= \sigma^2\, X^2(t)\, dt \end{aligned}$$

we obtain

$$d\ln(X(t)) = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma\, dB(t)$$

that integrated yields again to

$$X(t) = X(0)\, e^{\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma\, B(t)}.$$

2.2 The Uhlenbeck-Ornstein process

In this section we look for the solution, $X(t)$, of the following SDE

$$dX(t) = -\alpha\, X(t)\, dt + \sigma\, dB(t),$$

with α and σ two positive constants.

Use the test function $X(t) = f(t) = a(t)\left[c + \int_0^t b(s)\, dB(s)\right]$ whose differential is equal to

$$dX(t) = \frac{a'(t)}{a(t)} X(t)\, dt + a(t)\, b(t)\, dB(t).$$

Matching the coefficients of dt and $dB(t)$, we get

$$\frac{a'(t)}{a(t)} = -\alpha \quad \text{and} \quad a(t)\, b(t) = \sigma$$

that solved give

$$a(t) = a(0)\, e^{-\alpha\, t} \quad \text{and} \quad b(t) = \frac{\sigma}{a(t)} = \frac{\sigma}{a(0)} e^{\alpha\, t}$$

and setting $f(0) = X(0)$ we finally obtain

$$X(t) = X(0)\, e^{-\alpha\, t} + \sigma \int_0^t e^{-\alpha(t-s)}\, dB(s).$$

Using Proposition 3, we obtain that if $X(0)$ is independent of $B(t)$ and normally distributed (including the deterministic degenerate normal distribution), then $X(t)$ is a Gaussian process with mean and covariance functions

$$\begin{aligned} \mathbb{E}[X(t)] &= \mathbb{E}[X(0)]\, e^{-\alpha\, t} \to 0 \quad \text{as } t \to \infty \\ \mathbb{C}\text{ov}[X(t), X(s)] &= \mathbb{V}\text{ar}[X(0)]\, e^{-\alpha\,(t+s)} + \sigma^2 \int_0^{t \wedge s} e^{-\alpha(t+s-2u)}\, du \\ \mathbb{V}\text{ar}[X(t)] &= \mathbb{V}\text{ar}[X(0)]\, e^{-2\alpha\, t} + \sigma^2 \int_0^t e^{-2\alpha(t-u)}\, du \to \frac{\sigma^2}{2\alpha} \quad \text{as } t \to \infty \end{aligned}$$

Therefore we see that the O-U process admits a stationary distribution and we can construct a stationary version of the process by setting $X(0) \sim \mathrm{N}(0, \frac{\sigma^2}{2\alpha})$.

2.3 The Brownian Bridge

In this section we look for the solution, $X(t)$, of the following SDE

$$dX(t) = -\frac{X(t)}{1-t}\,dt + dB(t),$$

with $X(0) = 0$.

Use the test function $X(t) = f(t) = a(t)\left[c + \int_0^t b(s)\,dB(s)\right]$ whose differential is equal to

$$dX(t) = \frac{a'(t)}{a(t)} X(t)\,dt + a(t)\,b(t)\,dB(t).$$

Matching the coefficients of dt and $dB(t)$, we get

$$\frac{a'(t)}{a(t)} = -\frac{1}{1-t} \quad \text{and} \quad a(t)\,b(t) = 1$$

that solved give

$$a(t) = 1 - t \quad \text{and} \quad b(t) = \frac{1}{1-t}$$

and setting $X(0) = 0$ we finally obtain

$$X(t) = \int_0^t \frac{1-t}{1-s}\,dB(s). \tag{18}$$

Using Proposition 3, we obtain that $X(t)$ is a Gaussian process with mean and covariance functions

$$\mathbb{E}[X(t)] = 0 \quad \text{and} \quad \mathbb{C}\text{ov}[X(t), X(s)] = (1 - t \vee s)(t \wedge s)$$

that coincide with the ones of the Brownian Bridge. Since two Gaussian processes with identical mean and covariance functions are equal in distribution we see that equation (18) gives an alternative representation of the Brownian Bridge process.