Notes

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1 Brownian Motion with Drift

Definition 1. *The process $\{B_\mu(t), t \geq 0\}$ is called Brownian Motion with drift μ if it satisfies the following conditions*

i) $B_\mu(0) = 0$ *a.s.*

ii) It has independent and stationary increments

iii) $B_\mu(t) \sim \mathrm{N}(\mu\, t, t)$.

where $\mu \in \mathbb{R}$.

It is easy to realize that it can be written in the form

$$B_\mu(t) = \mu\, t + B(t)$$

where $B(t)$ is the standard Brownian Motion. If in addition we want to add a diffusion coefficient $\sigma > 0$ we could write the general (μ, σ)-BM, $W(t)$, in the following form

$$W(t) = \mu\, t + \sigma\, B(t).$$

1.1 Convergence of Simple Random Walks

It can be proved that we can approximate a (μ, σ)-BM, $W(t)$, by a sequence of Random Walks, $\{X_{\Delta t}(t), t \geq 0\}_{\Delta t > 0}$,

$$X_{\Delta t}(t) = \Delta x \sum_{n=1}^{\lfloor t/\Delta t \rfloor} X_n$$

with $\{X_n\}_{n \geq 0}$ being a sequence of iid. r.v. with the following distribution

$$X_n = \begin{cases} +1 & \text{w.p. } \frac{1}{2} + p(\Delta t) \\ \\ -1 & \text{w.p. } \frac{1}{2} - p(\Delta t) \end{cases}$$

where $\Delta x = \sigma \sqrt{\Delta t}$ and $p(\Delta t) = \frac{\mu}{2\sigma} \sqrt{\Delta t}$.

We have

$$X_{\Delta t}(t) \to_d W(t) \qquad \text{as } \Delta t \to 0.$$

1.2 Backward Diffusion Equations

Let $W(t)$ be a (μ, σ)-BM and define

$$p(x, t; y) = \lim_{\Delta x \to 0} \frac{\mathbb{P}\{x \leq W(t) \leq x + \Delta x | W(0) = y\}}{\Delta x}$$

the probability density to have $W(t) = x$ at time t conditioned on the event it started in y at time 0.

By conditioning on the position of the process at time $0 < h < t$, $W(h)$, we have that

$$\begin{aligned} p(x, t; y) &= \lim_{\Delta x \to 0} \int_{\mathbb{R}} \frac{\mathbb{P}\{x \leq W(t) \leq x + \Delta x | W(h) = z,\, W(0) = y\}}{\Delta x} \mathbb{P}\{W(h) \in dz | W(0) = y\} \\ &= \lim_{\Delta x \to 0} \int_{\mathbb{R}} \frac{\mathbb{P}\{x \leq W(t) \leq x + \Delta x | W(h) = z = y\}}{\Delta x} \mathbb{P}\{W(h) \in dz | W(0) = y\} \\ &= \int_{\mathbb{R}} p(x, t - h; y)\, \mathbb{P}_y\{W(h) \in dz\} \end{aligned}$$

where the second equality follows from the Markov property of the Brownian Motion and in the last we assumed we could exchange the integral and the limit operators.

Writing the integral as an expectation we have

$$p(x,t;y) = \mathbb{E}_y[p(x,t-h;W(h))],$$

or equivalently

$$p(x,t;y) - p(x,t-h;y) = \mathbb{E}_y[p(x,t-h;W(h)) - p(x,t-h;W(0))] = \mathbb{E}[g(y+Y) - g(y)], \tag{1}$$

where $g(y) = p(x,t-h;y)$ and $Y = W(h) - W(0)$. By Taylor expansion of $g(y)$ we have

$$g(y+\Delta y) = g(y) + g'(y)\,\Delta y + \frac{1}{2}g''(y)(\Delta y)^2 + \frac{1}{3!}g^{(3)}(\xi)(\Delta y)^3$$

with $\xi = \xi(\Delta y) \in [-\Delta y,\, \Delta y]$, and $Y \sim \mathrm{N}(\mu\, h, \sigma^2\, h)$ the following relations hold true

$$\mathbb{E}[Y] = \mu\, h \tag{2}$$

$$\mathbb{E}[Y^2] = \mathbb{V}\mathrm{ar}[Y] + \mathbb{E}^2[Y] = \sigma^2\, h + (\mu\, h)^2 = \sigma^2\, h + o(h) \tag{3}$$

$$\mathbb{E}[Y^3] = \mathbb{E}^3[Y] + 3\mathbb{V}\mathrm{ar}[Y]\,\mathbb{E}[Y] = h^3(\mu^3 + 3\,\mu\,\sigma^2) = o(h). \tag{4}$$

Therefore

$$\mathbb{E}[g(y+Y) - g(y)] = g'(y)\mathbb{E}[Y] + \frac{1}{2}g''(y)\mathbb{E}[Y]^2 + \frac{1}{3!}\mathbb{E}[g^{(3)}(\Xi)\,Y^3] \tag{5}$$

$$= g'(y)\mu\, h + \frac{1}{2}g''(y)\sigma^2\, h + o(h) \tag{6}$$

where $\Xi = \Xi(Y) \in [-Y,\, Y]$ and by equations (1) and (6), dividing by h and letting $h \to 0$ we finally get

$$\frac{\partial p}{\partial t}(x,t;y) = \mu\,\frac{\partial p}{\partial y}(x,t;y) + \frac{\sigma^2}{2}\frac{\partial^2 p}{\partial y^2}(x,t;y) \tag{7}$$

that is known as **Backward Diffusion Equation**.

1.3 Forward Diffusion Equations

Proceedings analogously to the previous section we can write the density transition function $p(x,t;y)$ in the following way

$$p(x,t;y) = \mathbb{E}[p(W(t-h), t-h; y) | W(t) = x],$$

obtaining

$$p(x,t;y) - p(x,t-h;y) = \mathbb{E}[p(W(t-h), t-h; y) - p(x,t-h;y) | W(t) = x] = \mathbb{E}[g(x-X) - g(x)], \tag{8}$$

where we have defined $g(x) = p(x,t-h;y)$ and $X = W(t) - W(t-h)$.

By doing the same steps as for the backward case we finally get

$$\frac{\partial p}{\partial t}(x,t;y) + \mu\,\frac{\partial p}{\partial x}(x,t;y) = \frac{\sigma^2}{2}\frac{\partial^2 p}{\partial x^2}(x,t;y) \tag{9}$$

that is known as **Forward Diffusion Equation**.

1.4 Solution of the Heat Equation

Consider the case of a $(0,\sigma)$-BM, say $B_\sigma(t)$. Then we already know that

$$(B_\sigma(t) | B_\sigma(0) = y) \sim \mathrm{N}(y, \sigma^2\, t),$$

and therefore the probability density at time t is given by

$$(B_\sigma(t) = x | B_\sigma(0) = y) = p(x,t;y) = \frac{1}{\sqrt{2\pi\,\sigma^2\,t}} e^{-\frac{(x-y)^2}{2\,\sigma^2\,t}},$$

that is solution of the following forward diffusion equation, also known as *heat differential equation*,

$$\frac{\partial p}{\partial t}(x,t;y) = \frac{\sigma^2}{2}\frac{\partial^2 p}{\partial x^2}(x,t;y) \tag{10}$$

where we used (9) with $\mu = 0$.

Therefore the solution of the heat equation with boundary condition

$$p(x,0;y) = \delta(x-y),$$

with $\delta(x)$ the Dirac delta function, is given by

$$p(x,t;y) = \frac{1}{\sqrt{2\pi\,\sigma^2\,t}} e^{-\frac{(x-y)^2}{2\,\sigma^2\,t}} = \int_{-\infty}^{+\infty} \delta(z-(x-y))\,\frac{1}{\sqrt{2\pi\,\sigma^2\,t}} e^{-\frac{z^2}{2\,\sigma^2\,t}} dz = \int_{-\infty}^{+\infty} \delta(z\,\sigma\sqrt{t} - (x-y))\,d\Phi(z)$$

where $\Phi(z) = \mathbb{P}\{Z \le z\}$ with $Z \sim \mathrm{N}(0,1)$ a standard Normal random variable.

In general if the starting position is distributed with density $p(y)$ then the density at time t, $p(x,t)$, is given by

$$\begin{aligned} p(x,t) = \mathbb{E}[p(x,t;Y)] &= \int_{-\infty}^{+\infty} p(x,t;y)\,p(y)\,dy \\ &= \int_{-\infty}^{+\infty} p(y)\,\frac{1}{\sqrt{2\pi\,\sigma^2\,t}} e^{-\frac{(x-y)^2}{2\,\sigma^2\,t}}\,dy = \int_{-\infty}^{+\infty} p(x - z\,\sigma\sqrt{t})\,d\Phi(z) \end{aligned}$$

with Y a r.v. having density $p(0,y)$.

Therefore we obtain the following result: the solution of the *heat equation*

$$h_t(x,t) = \frac{\sigma^2}{2}\,h_{xx}(x,t) \tag{11}$$

with initial boundary condition $h(x)$ is given by

$$h(x,t) = \int_{-\infty}^{+\infty} h(y)\,\frac{1}{\sqrt{2\,\pi\,\sigma^2\,t}} e^{-\frac{(x-y)^2}{2\sigma^2 t}}\,dy = \int_{-\infty}^{+\infty} h(x - z\,\sigma\sqrt{t})\,d\Phi(z) \tag{12}$$

1.5 Hitting Times

In this section we compute the probability for a Brownian motion with drift to hit a level $A > 0$ before the level $-B < 0$. Let write

$$p(y) = \mathbb{P}\{\text{hits } A \text{ before } -B | B_\mu(0) = y\} = \mathbb{P}_y\{\text{hits } A \text{ before } -B\}$$

for this probability. By defining $Y = B_\mu(h) - B_\mu(0)$ it is easy to realize that

$$p(y) = \mathbb{E}[p(y+Y)] + o(h)$$

and writing

$$p(y+Y) = p(y) + p'(y)\,Y + \frac{1}{2}p''(y)\,Y^2 + o(h)$$

we get

$$\mathbb{E}[p'(y)\,Y + \frac{1}{2}p''(y)\,Y^2] = o(h)$$

that divided by h and after letting $h \to 0$ gives

$$\mu\,p'(y) + \frac{1}{2}p''(y) = 0. \tag{13}$$

The solution of (13) with boundary conditions $p(A) = 1$ and $p(-B) = 0$ is given by

$$p(y) = \frac{e^{2\mu B} + e^{-2\mu x}}{e^{2\mu B} + e^{-2\mu A}}$$

and letting $\mu \to 0$ we get for the standard Brownian motion $B(t)$ that

$$p(y) = \frac{B + y}{B + A}.$$

2 Stochastic Calculus

In 1900, Bachelier proposed for the Paris stock exchange a model for the fluctuations affecting the price $X(t)$ of an asset that was given by the Brownian motion. By calling $dX(t)$ the infinitesimal variation of the price, he proposed

$$dX(t) = \mu dt + \sigma\, dB(t). \tag{14}$$

Even if at the moment equation (14) does not have a precise meaning, one can guess that it has as solution the (μ, σ)-BM, i.e.

$$X(t) = \mu\, t + \sigma\, B(t).$$

The problem of last solution is that

$$\mathbb{P}\{X(t) < 0\} > 0$$

especially when $\mu < 0$, where this probability increases to 1 as $t \to \infty$.

A solution to this modeling problem could be find by changing the absolute differential changing in the price $dX(t)$ by the relative changing, i.e. $\frac{dX(t)}{X(t)}$, in such a way to obtain the following Stochastic Differential Equation (SDE),

$$dX(t) = \mu\, X(t)dt + \sigma\, X(t)\, dB(t). \tag{15}$$

Again, at the moment we have no precise meaning for the SDE (15), but we could naively try to guess that the following steps will give the right solution to it.

$$\frac{dX(t)}{X(t)} = \mu\, dt + \sigma\, dB(t) \quad \Rightarrow \quad d\ln\{X(t)\} = \mu\, dt + \sigma\, dB(t) \quad \Rightarrow \quad X(t) = X(0)\, e^{\mu\, t + \sigma\, B(t)}. \quad \textbf{WRONG!!}$$

We will see after a precise definition of what we mean by a SDE that last guess is indeed a wrong guess and that more care we need when handling with stochastic calculus.

2.1 SDE seen as Stochastic Integral Equations

Before starting the definition of the stochastic integral we start noticing that it has no meaning to speak about the differential of the Brownian Motion, i.e. $dB(t)$. Indeed the Brownian motion has the property to have almost surely all the sample functions continuous but nowhere differentiable. To have just a felling of this let us look the following limit

$$\lim_{h \to 0} \frac{B(h) - B(0)}{\sqrt{h}} \overset{\mathcal{L}^2}{=} X \sim N(0, 1)$$

and from this we get that there is no defined limit for $(B(h) - B(0))/h$. Indeed for any $a > 0$ we have

$$\lim_{h \to 0} \mathbb{P}\{\frac{B(h) - B(0)}{h} > a\} = \lim_{h \to 0} \mathbb{P}\{\frac{B(h) - B(0)}{\sqrt{h}} > a\sqrt{h}\} = \frac{1}{2}$$

therefore if there were a limit r.v. it would take only values $\pm\infty$, each with probability 1/2.

Since we cannot speak about the differential Brownian Motion we need an alternative interpretation for the following SDE

$$dX(t) = a(t, X(t))dt + b(t, X(t))\, dB(t) \tag{16}$$

with $a(t, x)$ and $b(t, x)$ two given random functions.

The interpretation is the following, the stochastic process $X(t)$ is a solution of the SDE (16) if it satisfies for each $t \geq$ the following integral relation

$$X(t) = X(0) + \int_0^t a(s, X(s))ds + \int_0^t b(s, X(s))\, dB(s) \tag{17}$$

and we are left only with the definition of the meaning of the stochastic integral $\int_0^t b(s, X(s))\, dB(s)$.

2.2 Not uniqueness of the Riemann sum

A first attempt in the definition of the stochastic integral could be to define it as a limit of a Riemann sum. To this aim, given an interval $[0, t]$ let define $\Pi_n = \{0 = t_0 < t_1 < t_2 < \ldots < t_n = t\}$ be a partition of it in n sub intervals, and let define

$$|\Pi_n| = n, \quad ||\Pi_n|| = \max_{1 \leq k \leq n} (t_k - t_{k-1})\,.$$

In addition we denote by $\mathcal{P}_n$ the set of all partition of order n of the interval $[0, t]$ and by $\mathcal{P} = \bigcup_{n>0} \mathcal{P}_n$, the set of all partitions.

In general, if we have two continuous functions $f(t)$ and $g(t)$, mutually integrable (for this it is enough to have $g(t)$ of BV, see later for more details), the Riemann-Stieltjes integral is defined as

$$\int_0^t f(s) dg(s) = \lim_{||\Pi|| \to 0} \sum_{k=1}^{|\Pi|} f(\xi_k)[g(t_k) - g(t_{k-1})],$$

where ξ_k is any point in the interval $[t_{k-1}, t_k]$.

In the following we check that, as for the definition of the stochastic integral, it is crucial how to choose the sequence of points ξ_k, because to different choices correspond different values of the integral.

In the following we are going to compute $\int_0^t B(s)\, dB(s)$, with $B(t)$ the standard Brownian Motion, using two different extreme choices for ξ_k, i.e. $\xi_k = t_{k-1}$ and $\xi_k = t_k$.

Case 1: $\xi_k = t_{k-1}$

$$\int_0^t B(s)\, dB(s) = \lim_{||\Pi|| \to 0} \sum_{k=1}^{|\Pi|} B(t_{k-1})[B(t_k) - B(t_{k-1})].$$

Having that

$$B(t_k)B(t_{k-1}) = \frac{B^2(t_k) + B^2(t_{k-1})}{2} - \frac{[B(t_k) - B(t_{k-1})]^2}{2} \tag{18}$$

we obtain

$$\begin{aligned}
\int_0^t B(s)\, dB(s) &= \lim_{||\Pi|| \to 0} \left\{ \frac{1}{2} \sum_{k=1}^{|\Pi|} [B^2(t_k) - B^2(t_{k-1})] - \frac{1}{2} \sum_{k=1}^{|\Pi|} [B(t_k) - B(t_{k-1})]^2 \right\} \\
&= \frac{B^2(t)}{2} - \lim_{||\Pi|| \to 0} \frac{1}{2} \sum_{k=1}^{|\Pi|} [B(t_k) - B(t_{k-1})]^2 \stackrel{\mathcal{L}^2}{\to} \frac{B^2(t)}{2} - \frac{t}{2},
\end{aligned}$$

where we used the fact that

$$\sum_{k=1}^{|\Pi|} [B(t_k) - B(t_{k-1})]^2 \xrightarrow[||\Pi|| \downarrow 0]{\mathcal{L}^2} t, \tag{19}$$

to be proved later. Actually for the Brownian Motion, the limit in (19) is valid also with probability 1.

Case 2: $\xi_k = t_k$

$$\int_0^t B(s)\, dB(s) = \lim_{||\Pi|| \to 0} \sum_{k=1}^{|\Pi|} B(t_k)[B(t_k) - B(t_{k-1})].$$

Again using (18) we obtain

$$\int_0^t B(s)\,dB(s) = \lim_{||\Pi||\to 0} \left\{ \frac{1}{2}\sum_{k=1}^{|\Pi|}[B^2(t_k) - B^2(t_{k-1})] + \frac{1}{2}\sum_{k=1}^{|\Pi|}[B(t_k) - B(t_{k-1})]^2 \right\}$$
$$= \frac{B^2(t)}{2} + \lim_{||\Pi||\to 0} \frac{1}{2}\sum_{k=1}^{|\Pi|}[B(t_k) - B(t_{k-1})]^2 \overset{\mathcal{L}^2}{\to} \frac{B^2(t)}{2} + \frac{t}{2},$$

where we used again the convergence relation (19).

2.3 Definition of the Itô integral

From the previous section we have noticed that there could be some ambiguity in the definition of the stochastic integral via the limit of a Riemann sum, to avoid this we introduce the following definition.

Definition 2. *Given a standard Brownian Motion $\{B(t), t \geq 0\}$, we say that a stochastic process $\{X(t), t \geq 0\}$, is adapted to the filtration generated of the Brownian motion if*

$$1\{X(t) \in A\} = f(B(s), 0 \leq s \leq t),$$

for any event A.

The above definition just states that the process $X(t)$ cannot anticipate information about the Brownian Motion after time t. Given the definition of an adapted process we are now ready to define the Itô integral.

Definition 3. *Given the standard Brownian Motion $\{B(t), t \geq 0\}$ and an adapted stochastic process $\{X(t), t \geq 0\}$ satisfying the condition*

$$\int_0^t \mathbb{E}[X^2(s)]ds < \infty$$

the Itô integral, $I_t(X)$ is defined as

$$I_t(X) = \int_0^t X(s)\,dB(s) = \lim_{||\Pi||\to 0} \sum_{k=1}^{|\Pi|} X(t_{k-1})[B(t_k) - B(t_{k-1})]$$

where the limit is meant in $\mathcal{L}^2$ sense.