Notes

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1 The Renewal Equation

The renewal equation is an important integral equation that appears often in Renewal Theory. Its expression is given by

$$Z(t) = H(t) + \int_0^t Z(t-s)\, F(s), \quad t \geq 0 \tag{1}$$

with $Z(t)$, $H(t)$ and $F(t)$ general functions defined for $t \geq 0$. Usually the functions $F(t)$ and $H(t)$ are known and the task is to look for a function $Z(t)$ that satisfies equation (1) for all $t \geq 0$.

Remembering that the convolution $Z * F(t)$ of two functions defined on $\mathbb{R}^+$, $Z(t)$ and $F(t)$, is given by

$$Z * F(t) = \int_0^t Z(t-s)\, F(ds)$$

the renewal equation can be expressed also in the following alternative form

$$Z(t) = H(t) + Z * F(t).$$

In addition, since it is known that the Laplace transform of a convolution is given by the product of the Laplace transforms of the convoluted functions, i.e $\mathcal{L}[Z*F](s) = \mathcal{L}[Z](s) \cdot \mathcal{L}[F](s)$, equation (1) is equivalent to the following

$$\tilde{Z}(s) = \tilde{H}(s) + \tilde{Z}(s)\, \tilde{F}(s),$$

where we denoted by $\tilde{Z}(s) = \mathcal{L}[Z](s)$, $\tilde{F}(s) = \mathcal{L}[F](s)$ and $\tilde{H}(s) = \mathcal{L}[G](s)$ the Laplace transforms of $Z(t)$, $F(t)$ and $H(t)$ respectively.

Solving for $\tilde{Z}(s)$ we get that the Laplace transform of the solution function $Z(t)$ in (1) is given by

$$\tilde{Z}(s) = \frac{\tilde{H}(s)}{1 - \tilde{F}(s)}. \tag{2}$$

Since the Laplace transform uniquely characterizes the transformed function, the unique solution of the renewal equation (1) is given by the inverse transform of (2), i.e. $Z(t) = \mathcal{L}^{-1}[\tilde{Z}](t)$.

Noticing that

$$\frac{1}{1-\tilde{F}(s)} = \sum_{n=0}^{\infty} \tilde{F}(s)^n$$

we get directly that

$$Z(t) = \mathcal{L}^{-1}[\tilde{Z}](t) = \mathcal{L}^{-1}\left[\frac{\tilde{H}}{1-\tilde{F}}\right](t) = H * \sum_{n=0}^{\infty} F^{*n}(t), \tag{3}$$

where we denoted by $F^{*(n+1)}(t) = F^{*n} * F(t)$, for $n \geq 0$, having defined $F^{*0}(t) \equiv 1$.

Example 1. *Let $N(t)$ be a renewal process with general inter renewal time τ whose distribution is denoted by $F(t) = \mathbb{P}\{\tau \leq t\}$. It is known that the renewal function $m(t) = \mathbb{E}[N(t)]$ can be expressed in the following form*

$$m(t) = \sum_{n=1}^{\infty} F^{*n}(t) = F * \sum_{n=0}^{\infty} F^{*n}(t),$$

therefore it is solution of the following renewal equation

$$m(t) = F(t) + m * F(t).$$

Example 2. *Let $N_D(t)$ be a delayed renewal process with general inter renewal time τ whose distribution is denoted by $F(t) = \mathbb{P}\{\tau \leq t\}$ and with $H(t)$ being the distribution of the initial delay. Then the delayed renewal function $m_D(t) = \mathbb{E}[N_D(t)]$ has the following known expression*

$$m_D(t) = H * \sum_{n=1}^{\infty} F^{*n}(t),$$

therefore it is solution of the following renewal equation

$$m(t) = H(t) + m * F(t).$$

2 Ruin Probability in the Cramér-Lundberg risk model

Consider the Cramér-Lundberg risk model. It models the simplified dynamics of the risk reserve, $U(t)$, of an insurance company. It is assumed that the risk reserve increases linearly from the initial capital $U(0) = u$, thanks to the incoming premium flow rate, $c > 0$. In addition claims are supposed to come according to a homogeneous Poisson Process with parameter $\lambda > 0$ and the claims' sizes are assumed to be independent random variables, R_n, generally distributed with common distribution function $G(r) = \mathbb{P}\{R \leq r\}$.

Formally the model is described by the following relation

$$U(t) = U(0) + c\,t - \sum_{n=1}^{N(t)} R_n \tag{4}$$

where $N(t)$ denotes the Poisson counting process.

An important random variable associated to this model is the Ruin epoch, T^*, that is defined as the first moment the process assumes non-positive values, i.e.

$$T^* = \inf\{t \geq 0, U(t) < 0\}.$$

We assume that T^* takes value $+\infty$ over the event $\{U(t) \geq 0,\, t \geq 0\}$ that may have positive probability, and therefore it is interesting to ask for the probability, $p(u)$, of the set $\{T^* < \infty | U(0) = u\}$, that is called the Ruin probability under the assumption that the initial capital is equal to $u \geq 0$. In formulas

$$p(u) = \mathbb{P}\{T^* < \infty | U(0) = u\} = \mathbb{P}_u\{T^* < \infty\}.$$

In the following we compute the complement probability

$$Z(u) = 1 - p(u) = \mathbb{P}_u\{T^* = \infty\} = \mathbb{P}_u\{U(t) \geq 0,\, t \geq 0\},$$

that is the probability that the insurance company never incurs in ruin during its infinite lifetime.

Proposition 1. *The probability $Z(u)$ is solution of the following renewal equation*

$$Z(t) = Z(0) + Z * F(t) \tag{5}$$

where the function $F(t) = \frac{\lambda}{c} \int_0^t \bar{G}(r)\, dr$.

Proof. Conditioning on the pair (T_1, R_1), where T_1 is the time of occurrence of the first claim, and R_1 is its size we have that

$$Z(u) = \mathbb{P}_u\{U(t) \geq 0,\, t \geq 0\} = \int_0^{\infty} \int_0^{\infty} \mathbb{P}_u\{U(t) \geq 0,\, t \geq 0 | (T_1, R_1) = (s, r)\}\, \lambda\, e^{-\lambda\, s}\, ds\, G(dr), \tag{6}$$

where we used the fact that the first arrival time T_1 is equal to the first inter arrival time and, as such, exponentially distributed with parameter $\lambda > 0$. The behavior of the risk reserve process can be described in the following way

$$\begin{aligned} U(t) &= U(0) + c\,t & 0 \leq t < T_1 \\ U(T_1) &= U(0) + c\,T_1 - R_1 & \\ U(t + T_1) &= U(T_1) + c\,t - \sum_{n=2}^{N(t+T_1)} R_n & t \geq 0. \end{aligned}$$

Noticing that

$$\{N(t+T_1)-1\}_{t\geq 0} =_d \{N(t)\}_{t\geq 0}$$

we immediately have that

$$\{U(t+T_1)|U(T_1)=u\}_{t\geq 0} =_d \{U(t)|U(0)=u\}_{t\geq 0}$$

for any $u\in\mathbb{R}$, where $=_d$ denotes equality in distribution.

Assuming that $u\geq 0$, w.p.1 $U(t)\geq 0$ for $0\leq t<T_1$, and therefore it follows that

$$\begin{aligned}
\mathbb{P}_u\{U(t)\geq 0,\, t\geq 0|(T_1,R_1)=(s,r)\} &= \mathbb{P}_u\{U(t+T_1)\geq 0,\, t\geq 0|(T_1,R_1)=(s,r)\}\\
&= \mathbb{P}_u\{U(t+T_1)\geq 0,\, t\geq 0|U(T_1)\geq 0,(T_1,R_1)=(s,r)\}\\
&\quad\cdot \mathbb{P}_u\{U(T_1)\geq 0|(T_1,R_1)=(s,r)\}\\
&= \mathbb{P}_{u+c\,s-r}\{U(t)\geq 0,\, t\geq 0\}\,1\{u+c\,s-r\geq 0\}\\
&= Z(u+c\,s-r)\,1\{u+c\,s-r\geq 0\},
\end{aligned}$$

that substituted in (6) gives

$$\begin{aligned}
Z(u) &= \int_0^\infty\int_0^\infty Z(u+c\,s-r)\,1\{u+c\,s-r\geq 0\}\,\lambda\,e^{-\lambda\,s}\,ds\,G(dr)\\
&= \int_0^\infty\int_0^{u+c\,s} Z(u+c\,s-r)\,\lambda\,e^{-\lambda\,s}\,ds\,G(dr)\\
&= \int_0^\infty \lambda\,e^{-\lambda\,s}\left[\int_0^{u+c\,s} Z(u+c\,s-r)\ G(dr)\right]ds
\end{aligned}$$

using the change of variable $t=u+c\,s$, we get

$$= \int_u^\infty \frac{\lambda}{c}\,e^{-\lambda\,\frac{t-u}{c}}\left[\int_0^t Z(t-r)\,G(dr)\right]dt$$

Multiplying both sizes for $e^{-\frac{\lambda}{c}u}$ we finally get

$$Z(u)\,e^{-\frac{\lambda}{c}u} = \frac{\lambda}{c}\int_u^\infty e^{-\frac{\lambda}{c}t}\left[\int_0^t Z(t-r)\,G(dr)\right]dt$$

that shows that $Z(u)$ is differentiable and indeed deriving it with respect to u and simplifying, we get

$$Z'(u) = \frac{\lambda}{c}Z(u) - \frac{\lambda}{c}\int_0^u Z(u-r)\,G(dr) = \frac{\lambda}{c}Z(u) + \frac{\lambda}{c}\int_0^u Z(u-r)\,\bar{G}(dr) \tag{7}$$

Integrating equation (7) on $[0,t]$ we get

$$Z(t)-Z(0) = \int_0^t Z'(u)\,du = \frac{\lambda}{c}\int_0^t Z(u)\,du + \frac{\lambda}{c}\int_0^t\int_0^u Z(u-r)\,\bar{G}(dr)\,du \tag{8}$$

and setting

$$h(r) = \begin{cases} \int_0^{t-r} Z(u)du & r\leq t\\ 0 & r\geq t\end{cases}$$

we can rewrite equation (8) in the following way

$$\begin{aligned}
Z(t)-Z(0) &= \frac{\lambda}{c}h(0) + \frac{\lambda}{c}\int_0^t h(r)\,\bar{G}(dr)\\
&= \frac{\lambda}{c}h(0) + \frac{\lambda}{c}h(r)\,\bar{G}(r)\Big|_0^t - \frac{\lambda}{c}\int_0^t \bar{G}(r)\,h'(r)\,dr\\
&= -\frac{\lambda}{c}\int_0^t \bar{G}(r)\,h'(r)\,dr
\end{aligned}$$

and having that $h'(r) = -Z(t-r)\,1\{r \le t\}$ we finally obtain

$$Z(t) = Z(0) + \frac{\lambda}{c}\int_0^t Z(t-r)\,\bar{G}(r)\,dr$$

from that the result follows. $\square$

Proposition 2. *Assuming $c > \lambda\,\rho$, the ruin probability $p(u)$ is given by*

$$p(u) = 1 - (1-p(0))\sum_{n=0}^{\infty} F^{*n}(t), \tag{9}$$

where the function $F(t) = \frac{\lambda}{c}\int_0^t \bar{G}(r)\,dr$, $\rho = \mathbb{E}[R] = \int_0^\infty \bar{G}(r)\,dr$ and

$$p(0) = \frac{\lambda\,\rho}{c}. \tag{10}$$

Proof. From (3) and (5) with $H(t) \equiv Z(0)$ we immediately get

$$Z(u) = Z(0)\sum_{n=0}^{\infty} F^{*n}(t).$$

Therefore we need only to compute $Z(0)$. Applying the theorem on the Renewal Reward processes it follows that

$$\lim_{t\to\infty} \frac{c\,t - \sum_{n=1}^{N(t)} R_n}{t} = c - \lambda\,\rho$$

and having $c > \lambda\,\rho$, we obtain that

$$c\,t - \sum_{n=1}^{N(t)} R_n \to +\infty \quad \text{a.s.}$$

Therefore $m = \inf\{c\,t - \sum_{n=1}^{N(t)} R_n, t \ge 0\}$ is a well defined random variable and

$$Z(u) = \mathbb{P}\{u + m \ge 0\} = \mathbb{P}\{A_u\}$$

where $A_u = \{\omega \in \Omega : m(\omega) \ge -u\}$.

Since $A_{u'} \subset A_{u''}$ when $u' \le u''$, we have that A_u is an increasing sequence of sets and from the continuity property $(*)$ of the probability

$$Z(\infty) = \lim_{u\uparrow\infty} Z(u) = \lim_{u\uparrow\infty} \mathbb{P}\{A_u\} \overset{*}{=} \mathbb{P}\{\lim_{u\uparrow\infty} A_u\} = \mathbb{P}\{\Omega\} = 1.$$

Therefore using bounded convergence theorem in (5) we have that

$$1 = Z(0) + F(\infty)$$

because $Z(t) \to 1$ as $t \to \infty$ and therefore $Z * F(t) \to F(\infty)$ as $t \to \infty$ with $F(\infty) = \lambda\,\rho/c$.

Since $p(0) = 1 - Z(0)$ the result follows. $\square$

3 Waiting time distribution in a $M/G/1$ queue

Let W_n denote the waiting time of the customer n that arrives at time T_n, departs at time D_n and whose service time is σ_n. Call also $\tau_n = T_n - T_{n-1}$ the inter-arrival time between the $(n-1)$-th and the n-th customers.

Obviously $D_n = T_n + W_n + \sigma_n$ and the waiting time of the $(n+1)$-th customer is given by

$$W_{n+1} = \begin{cases} D_n - T_{n+1} & \text{if } T_{n+1} \le D_n \\ 0 & \text{if } T_{n+1} > D_n \end{cases} \tag{11}$$

that can be shortly rewritten as

$$W_{n+1} = (D_n - T_{n+1})^+ = (W_n + \sigma_n + T_n - T_{n+1})^+ = (W_n + \sigma_n - \tau_{n+1})^+ = (W_n + X_n)^+ \tag{12}$$

with $(a)^+ = \max\{a, 0\}$ and $X_n = \sigma_n - \tau_{n+1}$. Obviously the sequence $\{X_n\}_{n\geq 0}$ are iid and we assume that at time 0 there is a customer entering in the system with service request $\sigma_0 \geq 0$ and waiting time $W_0 \geq 0$.

Letting $S_0 = 0$ and $S_n = \sum_{k=0}^{n-1} X_k$, for $n > 0$, be the free random walk, we have the following important result

Proposition 3. *The solution of the so called* ***Lindley's recursion rule***

$$W_{n+1} = (W_n + X_n)^+, \quad n \geq 0 \tag{13}$$

is equal in distribution to

$$W_{n+1} =_d \max\{W_0 + S_{n+1}, M_n\} \tag{14}$$

where $=_d$ *means equality in distribution and* $M_n = \max\{S_k, 0 \leq k \leq n\}$.

Proof. First we prove that

$$W_{n+1} = \max\{W_0 + S_{n+1}, S_{n+1} - S_1, S_{n+1} - S_2, \dots, S_{n+1} - S_n, 0\}. \tag{15}$$

Indeed last equation is valid for $n = 0$ since

$$W_1 = (W_0 + X_0)^+ = (W_0 + S_1)^+ = \max\{W_0 + S_1, 0\}.$$

Now assuming that it is valid for $n \geq 0$ we have that

$$\begin{aligned} W_{n+2} &= (W_{n+1} + X_{n+1})^+ \\ &= (\max\{W_0 + S_{n+1}, S_{n+1} - S_1, S_{n+1} - S_2, \dots, S_{n+1} - S_n, 0\} + X_{n+1})^+ \\ &= (\max\{W_0 + S_{n+1} + X_{n+1}, S_{n+1} + X_{n+1} - S_1, S_{n+1} + X_{n+1} - S_2, \dots, S_{n+1} + X_{n+1} - S_n, X_{n+1}\})^+ \\ &= (\max\{W_0 + S_{n+2}, S_{n+2} - S_1, S_{n+2} - S_2, \dots, S_{n+2} - S_n, S_{n+2} - S_{n+1}\})^+ \\ &= \max\{W_0 + S_{n+2}, S_{n+2} - S_1, S_{n+2} - S_2, \dots, S_{n+2} - S_n, S_{n+2} - S_{n+1}, 0\}. \end{aligned}$$

In addition we have that

$$\begin{aligned} (S_{n+1} - S_1, S_{n+1} - S_2, \dots, S_{n+1} - S_n, 0) &= \left(\sum_{k=1}^{n} X_k, \sum_{k=2}^{n} X_k, \dots, \sum_{k=n}^{n} X_k, 0\right) \\ &=_d \left(\sum_{k=0}^{n-1} X_k, \sum_{k=0}^{n-2} X_k, \dots, \sum_{k=0}^{0} X_k, 0\right) = (S_n, S_{n-1}, \dots, S_1, S_0) \end{aligned}$$

and therefore

$$\max\{S_{n+1} - S_1, S_{n+1} - S_2, \dots, S_{n+1} - S_n, 0\} =_d M_n. \tag{16}$$

From (15) and (16) we finally get

$$W_{n+1} =_d \max\{W_0 + S_{n+1}, M_n\}$$

and the result follows. □

Proposition 4. *Assuming that* $\mathbb{E}[X] < 0$ *in the limit as* $n \to \infty$ *the stationary distribution of the waiting time will be distributed as* $M = sup_{n\geq 0}\{S_n\}$*, i.e.*

$$W_\infty =_d M.$$

Proof. From the LLN

$$\lim_{n\to\infty} \frac{S_n}{n} = \mathbb{E}[X] < 0$$

we get that $S_n \to -\infty$ a.s. as $n \to \infty$, and therefore M is a well defined random variable. In addition $W_0 + S_n \to -\infty$ as well and therefore $\max\{W_0 + S_n, M_n\} \sim M_n \to M$ as $n \to \infty$, that gives the result. □

3.1 Relation between the waiting time distribution and the ruin probability

Proposition 5. *Assuming that the general interval between two customers τ is distributed as $c\,\tau^*$, with τ^* the general interval between two claims and that the general claim size, R, is distributed as the general service time σ, then*

$$\mathbb{P}\{W_\infty \le u\} = 1 - p(u).$$

Proof. From the definition of $Z(u)$ we have that

$$\begin{aligned} Z(u) = \mathbb{P}_u\{U(t) \ge 0,\, t \ge 0\} &= \mathbb{P}_u\{\inf_{t\ge 0} U(0) + c\,t - \sum_{n=1}^{N(t)} R_n \ge 0\} = \mathbb{P}\{\inf_{t\ge 0} c\,t - \sum_{n=1}^{N(t)} R_n \ge -u\} \\ &= \mathbb{P}\{-\inf_{t\ge 0} c\,t - \sum_{n=1}^{N(t)} R_n \le u\} = \mathbb{P}\{\sup_{t\ge 0} \sum_{n=1}^{N(t)} R_n - c\,t \le u\} \\ &= \mathbb{P}\{\sup_{n\ge 1}(\sum_{k=1}^{n} R_k - c\,T_k)^+ \le u\} = \mathbb{P}\{\sup_{n\ge 1}(\sum_{k=1}^{n}(R_k - c\,\tau_k^*))^+ \le u\} \\ &= \mathbb{P}\{\sup_{n\ge 1} S_n \le u\} = \mathbb{P}\{M \le u\} \end{aligned}$$

where $M = \sup_{n\ge 0} S_n$, with $S_n = \sum_{k=1}^{n}(R_k - c\,\tau_k^*)$ and $S_0 = 0$.

The result follows applying Proposition 3 having $R_n - c\,\tau_n^* =_d \sigma_n - \tau_{n+1}$ and $p(u) = 1 - Z(u)$. □