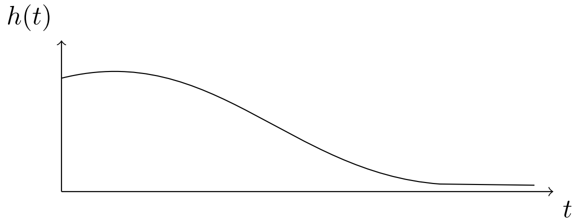
Comments

March 11^{th}, 2010

1 The Key Renewal Theorem - interpretation

Let assume that we have a renewal process $N(t)$, i.e. given a sequence of positive iid random variables $\{\tau_n\}_{n\geq 1}$ with common distribution $F(t) = \Pr\{\tau \leq t\}$, $T_n = \sum_{k=1}^{n} \tau_k$, for $n \geq 1$ and $T_0 = 0$, $N(t) = \max_n\{T_n \leq t\}$.

Figure 1: Impulse function.



Assume that each time a renewal occur, an impulse is generated, where the impulse function $h(t)$ is a general directly Riemann function as shown in Figure 1. The effect at time $t \geq T_n$ of an impulse generated at the n-th renewal occurrence is given by $h(t - T_n)$, therefore if we consider the total effect of all impulses generated before time t, say $H(t)$, we get that it is equal to

$$H(t) = \sum_{n=1}^{N(t)} h(t - T_n).$$

Since $N(t)$ is a non decreasing function, for $t \geq 0$ we can consider it as a positive random measure on $\mathbb{R}$ that locates a unit dirac mass function at any renewal epoch, so that

$$N(t) = \sum_{n=1}^{\infty} 1\{T_n \leq t\}; \quad N(dt) = dN(t) = \sum_{n=1}^{\infty} \delta_{T_n}(dt).$$

Using this interpretation it is possible to represent the total impulse at time t in the following alternative way

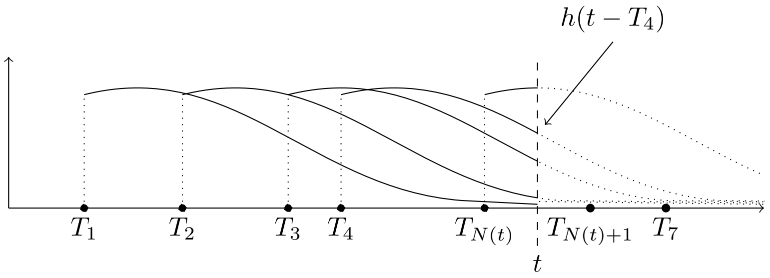
$$H(t) = \int_0^t h(t - s) dN(s); \tag{1}$$

Figure 2 shows a graphical representation of $H(t)$.

After a changing of variable, the integral in equation (1) may be rewritten also in the following form

$$H(t) = \int_0^t h(s) dN(t - s) = \sum_{n=-N(t)}^{-1} h(T_n); \tag{2}$$

Figure 2: Joint contributions of impulses at time t.



where we have defined T_{-n}, with $n > 0$, as the n-th event before t, i.e. $T_{-1} = T_{N(t)}$, $T_{-2} = T_{N(t)-1}$, $\ldots$, $T_{-N(t)} = T_1$.

Letting now $t \to \infty$, we have that the renewal process reaches the stationary regime and equation (2) informally becomes

$$H(\infty) = \int_0^\infty h(s) dN^*(-s) = \sum_{n=-\infty}^{-1} h(T_n^*), \tag{3}$$

where we denoted by T_{-n}^*, $n > 0$, the n-th arrival before 0 of the stationary renewal process $N^*(t)$, $t \in (-\infty, \infty)$.

Calling $Y(t) = T_{N(t)+1} - t$, and $A(t) = t - T_{N(t)}$, respectively the residual lifetime and the age of the renewal process, we know that

$$\lim_{t\to\infty} \Pr\{Y(t) \leq x\} = \lim_{t\to\infty} \Pr\{A(t) \leq x\}$$

implying that in the limit $N^*(-t) =_d N^*(t)$, $t \geq 0$ and therefore

$$H(\infty) = \int_0^\infty h(s) dN^*(s) = \sum_{n=1}^{\infty} h(T_n^*). \tag{4}$$

Noticing that the renewal function for a stationary renewal process is equal to $m^*(t) = \mathbb{E}[N^*(t)] = \lambda\, t$, for $t \geq 0$ with $\lambda = \mathbb{E}[\tau]^{-1}$, we get

$$\begin{aligned}\mathbb{E}[H(\infty)] = \mathbb{E}\left[\int_0^\infty h(s) dN^*(s)\right] &= \int_0^\infty h(s)\mathbb{E}[dN^*(s)] \\ &= \int_0^\infty h(s) m^*(ds) = \lambda \int_0^\infty h(s) ds\end{aligned}$$

Taking expectation in formula (1)

$$\mathbb{E}[H(t)] = \mathbb{E}\left[\int_0^t h(t-s) dN(s)\right] = \int_0^t h(t-s)\mathbb{E}[dN(s)] = \int_0^t h(t-s) m(ds)$$

and assuming that $\mathbb{E}[H(t)] \to \mathbb{E}[H(\infty)]$ as $t \to \infty$, we finally get

$$\lim_{t\to\infty} \int_0^t h(t-s) m(ds) = \lambda \int_0^\infty h(s) ds, \tag{5}$$

that is the **Key Renewal Theorem** (**KRT**).

2 Equivalence of Blackwell's Theorem and the Key Renewal Theorem.

The Blackwell's Theorem states that the renewal function $m(t)$ satisfies the following limit equation

$$\lim_{t\to\infty} m(t+a) - m(t) = \lambda\, a, \quad a \geq 0 \tag{6}$$

where $\lambda = \mathbb{E}[\tau]^{-1}$, with τ the general inter-renewal interval.

If we consider the function

$$h(t) = 1\{a \leq t \leq b\}, \quad \text{with}\, 0 \leq a \leq b < \infty$$

we have that $\int_0^\infty h(t)dt = b - a$ and

$$\begin{aligned}\int_0^t h(t-s)\, m(ds) &= \int_0^\infty 1\{a \leq t-s \leq b\} 1\{s \leq t\}\, m(ds) \\ &= \int_0^\infty 1\{t-b \leq s \leq t-a\}\, m(ds) \\ &= \int_{t-b}^{t-a} m(ds) = m(t-a) - m(t-b).\end{aligned}$$

From last equation we directly get that the Key Renewal Theorem implies the Blackwell's Theorem, indeed we have

$$\begin{aligned}\lim_{t\to\infty} m(t+(b-a)) - m(t) &= \lim_{t\to\infty} m(t-a) - m(t-b) \\ &= \lim_{t\to\infty} \int_0^t h(t-s)\, m(ds) \overset{KRT}{=} \lambda \int_0^\infty h(t)dt = \lambda(b-a)\end{aligned}$$

for any $b - a \geq 0$.

To see the opposite implication we get that

$$\begin{aligned}\lambda \int_0^\infty h(t)dt = \lambda(b-a) &\overset{BwT}{=} \lim_{t\to\infty} m(t+(b-a)) - m(t) \\ &= \lim_{t\to\infty} m(t-a)) - m(t-b) = \lim_{t\to\infty} \int_0^t h(t-s)\, m(ds)\end{aligned}$$

for any $0 \leq a \leq b < \infty$.

Last relation is valid then for indicator functions, and obviously linear combination of them, therefore it is valid for simple functions with bounded support.

The general result then follows by splitting the general directly Riemann integrable function $h(t) = h^+(t) + h^-(t)$ in its positive part, $h^+(t) = \max(h(t), 0)$ and its negative part $h^-(t) = \min(h(t), 0)$ and then noticing that any positive directly Riemann integrable function $f(t)$ can be approximated from below by a sequence of simple functions $f_n(t)$ of bounded support whose expression is the following

$$f_n(t) = \sum_{k=0}^{n\, 2^n - 1} \inf \left\{ f(t),\, k\, 2^{-n} \leq t < (k+1)2^{-n} \right\} 1\{k\, 2^{-n} \leq t < (k+1)2^{-n}\}.$$