Comments

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Proposition 1. *Let $N(t)$ be a random process such that*

$$N(t) \to \infty \quad \text{w.p.1}, \tag{1}$$

as $t \to \infty$ and where "w.p.1" stays for "with probability 1". In addition let S_n be a sequence of random variable such that

$$S_n \to \mu \quad \text{w.p.1}, \tag{2}$$

as $n \to \infty$.

Then we have that also

$$S_N(t) \to \mu \quad \text{w.p.1}. \tag{3}$$

Proof. From (1) we have that $\forall \omega \in \Omega$

$$\lim_{t\to\infty} N(t,\omega) = \infty,$$

and from (2) we have that $\forall \omega \in \Omega$

$$\lim_{n\to\infty} S(n,\omega) = \mu.$$

Then by changing of variable, $\forall \omega \in \Omega$

$$\lim_{t\to\infty} S(N(t,\omega),\omega) = \lim_{n_t = N(t,\omega)\to\infty} S(n_t,\omega) = \mu.$$

□

Proposition 2. *Given a positive integer-valued random variable N, with distribution function $F(t)$, for $t \geq 0$, we can compute its mean in the following way*

$$\mathbb{E}[N] = \int_0^\infty \bar{F}(t)dt = \sum_{n=0}^\infty \bar{F}(n) = \sum_{n=1}^\infty \Pr\{N \geq n\},$$

where $\bar{F}(t) = 1 - F(t) = \Pr\{N > n\}$ is called the tail distribution

Proof. First notice that the three ways to write the expectation are equivalent. Indeed we have

$$\int_0^\infty \bar{F}(t)dt = \sum_{n=0}^\infty \int_n^{n+1} \bar{F}(t)dt = \sum_{n=0}^\infty \bar{F}(n) \int_n^{n+1} dt = \sum_{n=0}^\infty \bar{F}(n),$$

where in the second equality we used the fact that for an integer-valued random variable $\bar{F}(t) = \bar{F}(\lfloor t \rfloor)$. In addition

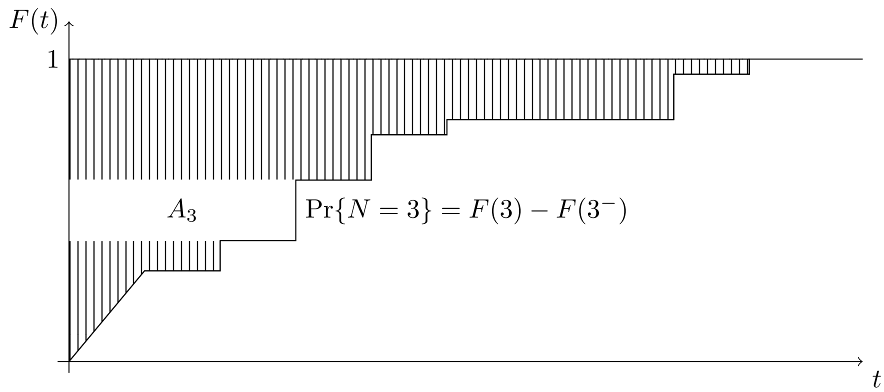
$$\sum_{n=0}^\infty \bar{F}(n) = \sum_{n=0}^\infty \Pr\{N > n\} = \sum_{n=0}^\infty \Pr\{N \geq n+1\} = \sum_{n=1}^\infty \Pr\{N \geq n\}$$

To verify that these quantities are equal to the expected value just notice that

$$\begin{aligned}\mathbb{E}[N] &= \sum_{n=0}^{\infty} n \Pr\{N=n\} = \sum_{n=1}^{\infty} n \Pr\{N=n\} = \sum_{n=1}^{\infty} n\, \mathbb{E}[1\{N=n\}] \\ &= \mathbb{E}\Big[\sum_{n=1}^{\infty} n\, 1\{N=n\}\Big] = \mathbb{E}\Big[\sum_{n=1}^{\infty} 1\{N \geq n\}\Big] \\ &= \sum_{n=1}^{\infty} \mathbb{E}[1\{N \geq n\}] = \sum_{n=1}^{\infty} \Pr\{N \geq n\}\end{aligned}$$

All the exchange of the infinite sums with the expectation are justified by the monotone convergence theorem. Another way to see the validity of the formula

Figure 1: Graphical interpretation of the expectation



is to look at the Figure (1) and realize that

$$\mathbb{E}[N] = \sum_{n=0}^{\infty} n \Pr\{N=n\} = \sum_{n=1}^{\infty} A_n$$

where A_n is the area of the rectangle with base n and altitude $\Pr\{N=n\}$. Summing all these areas we get exactly the total area below the curve given by the tail distribution $\bar{F}(t)$ that is

$$\mathbb{E}[N] = \int_0^{\infty} \bar{F}(t) dt.$$

□