$$\sum_{k=0}^{2n+1} \frac{(-x)^k}{k!} \le e^{-x} \le \sum_{k=0}^{2n} \frac{(-x)^k}{k!}$$

$$e^{-\theta x} - 1 + \theta x \le \frac{(\theta x)^2}{2}$$

$\int_0^{\varepsilon} x^2 d\mu_f(x) < \infty.$

$$\Rightarrow e^{\int_{(0,\varepsilon]} (e^{-\theta x} - 1 + \theta x)} \le e^{\frac{\theta^2}{2} \int_{(0,\varepsilon]} x^2 d\mu_f} \xrightarrow[\theta \downarrow 0]{} 1$$

↳ This gives an idea to understand that the sum does converge to a well defined r.v.

* $Y = \int_{(0,\varepsilon)} x\, d(N_f - \mu_f)$ is a well defined r.v.

with $\mathbb{E}\{e^{-\theta Y}\} = e^{\int_{(0,\varepsilon]} (e^{-\theta x} - 1 + \theta x)\, d\mu_f(x)}$

* $Z = \int_{(\varepsilon,\infty)} x\, dN_f$ $\quad Y \perp Z$

$\mathbb{E}(e^{-\theta Z}) = e^{\int_{(\varepsilon,\infty)} (e^{-\theta x} - 1)\, d\mu_f(x)}$

* $\mathbb{E}\{e^{-\theta(Y+Z)}\} = \mathbb{E}\{e^{-\theta Y}\}\, \mathbb{E}\{e^{-\theta Z}\}$

$= e^{\int_{(\varepsilon,\infty)} (e^{-\theta x} - 1)\, d\mu_f(x)}\, e^{\int_{(0,\varepsilon)} (e^{-\theta x} - 1 + \theta x)\, d\mu_f}$

$\mathbb{E}\{e^{-\theta(\int_{(0,\varepsilon]} x\, d(N_f - \mu_f) + \int_{(\varepsilon,\infty)} x\, dN_f)}$

$= e^{\int (e^{-\theta x} - 1 + \theta x\, \mathbb{I}_{(0,\varepsilon]}(x))\, d\mu_f(x)}.$

We assumed that $f \ge 0$.

In general, when $f: \bigcirc \to \mathbb{R}$,

$\mathbb{E}\{e^{i\theta(\int_{[-\varepsilon,\varepsilon]} x\, d(N_f - \mu_f) + \int_{[-\varepsilon,\varepsilon]^c} x\, dN_f)}\}$

$= e^{\int (e^{i\theta x} - 1 + i\theta x\, \mathbb{I}_{[-\varepsilon,\varepsilon]}(x))\, d\mu_f(x)}.$

Provided $\mu([-\varepsilon,\varepsilon]^c) < \infty$, $\int_{[-\varepsilon,\varepsilon]} x^2 d\mu_f(x) < \infty$

$\int (x^2 \wedge \varepsilon)\, d\mu_f(x) < 1$

$\int (f^2 \wedge 1)\, d\mu < \infty$

* Let N_f be a Poisson random measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu_f)$,

$N_{f,1}, \ldots, N_{f,n}$ indep. Poisson random measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \frac{1}{n}\mu_f)$.

Now, we take $M_n = \sum_{i=1}^n N_{f,i}$ ← Poisson random measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu_f)$.

Now, we make

$$Y_i = \int_{(0,\varepsilon]} x \, d(N_{f,i} - \tfrac{1}{n}\mu_f) + \int_{(\varepsilon,\infty)} x \, dN_{f,i}$$

$\Rightarrow$ $\{Y_i\}_{i=1}^n$ indep r.v.

Also, $Y_1, \dots, Y_n$ iid, and $\sum_{i=1}^n Y_i = \int_{(0,\varepsilon]} x \, d(N_f - \mu_f) + \int_{(\varepsilon,\infty)} x \, dN_f$

$\uparrow$ This is true $\forall n \in \mathbb{N}$.

$\uparrow$ This kind of property is called

$\boxed{\text{Infinite divisible distribution}}$

* We see that $Y+Z$ has an infinite divisible distribution.

Def $\Rightarrow$ X has an infinite divisible distribution if $\forall n \in \mathbb{N}$ $\exists \{X_i^n\}_{i=1}^n$ iid s.t. $X = \sum_{i=1}^n X_i^n$

Examples $\Rightarrow$ Gamma, normal, Cauchy, Poisson ...

* $X \perp Y, Z$, $X \sim N(\mu, \sigma^2)$

$$\mathbb{E}(e^{\theta X}) = e^{\theta\mu + \frac{\sigma^2}{2}\theta^2}, \quad \mathbb{E}(e^{i\theta X}) = e^{i\theta\mu - \frac{\sigma^2}{2}\theta^2}$$

* You can build Gamma, Cauchy and Poisson from a Poisson random measure. With the normal, you can't

* $\nu(A) = \mu_f(A) - \mu_f(\{0\} \cap A)$

$$= \begin{cases} \mu_f(A) - \mu_f(\{0\}) & 0 \in A \\ \mu_f(A) & \text{o.w.} \end{cases}$$

- $\int_{\mathbb{R}} (x^2 \wedge 1) \, d\nu(x) < \infty$
- $\mathbb{E}\{e^{i\theta(Y+Z)}\} = e^{\int (e^{i\theta x} - 1 - i\theta x \, \mathbb{I}_{[\varepsilon,\varepsilon]}(x)) \, d\nu(x)}$

$$\mathbb{E}\{e^{i\theta X}\} = e^{i\theta\mu - \frac{\sigma^2}{2}\theta^2}$$

$$\Rightarrow \mathbb{E}\{e^{i\theta(X+Y+Z)}\} = \exp\left\{ i\theta\mu - \frac{\sigma^2}{2}\theta^2 + \int (e^{i\theta x} - 1 + i\theta x \, \mathbb{I}_{[-\varepsilon,\varepsilon]}(x)) \, d\nu(x) \right\} \quad (*)$$

$\nu(\{0\}) = 0$, $\int (x^2 \wedge 1) \, d\nu(x) < \infty$

* $X+Y+Z$ has an infinitely divisible distribution.

Theorem $\Rightarrow$ A r.v. has an infinite divisible distribution $\Leftrightarrow$ it can be constructed as above. That is, for any such r.v., there is a unique measure ν on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with $\nu(\{0\}) = 0$, $\int (x^2 \wedge 1) \, d\nu(x) < \infty$, called the $\boxed{\text{Lévy measure}}$, μ, σ^2, s.t. its characteristic function is given by $(*)$. Thus, it can be represented as a sum of a normal r.v. and another r.v. obtained from a Poisson random measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \nu)$ as we did.

γ, σ^2!

$\mu = \mu(\varepsilon)$

* $X \sim \text{Poisson}(\lambda)$

$\mathbb{E}(e^{i\theta X}) = e^{\lambda(e^{i\theta}-1)}$; $\mathbb{E}(e^{-\theta X}) = e^{-\lambda(1-e^{-\theta})}$

$\cdot\, i\theta\mu - \frac{\sigma^2}{2}\theta^2 + \int (e^{i\theta x} - 1 - i\theta x\, \mathbb{I}_{[-\varepsilon,\varepsilon]}(x))\, d\nu(x)$

$= i\theta \{\mu - \int_{[-\varepsilon,\varepsilon]} x\, d\nu(x)\} + \int (e^{i\theta x} - 1)\, d\nu(x) - \frac{\sigma^2}{2}\theta^2$

* $i\theta\mu - \frac{\sigma^2}{2}\theta^2 + \int (e^{i\theta x} - 1 - i\theta x\, \mathbb{I}_{[-\varepsilon,\varepsilon]}(x))\, d\nu(x)$; $\int (x^2 \wedge 1)\, d\nu(x) < \infty$

↑ such a measure always $\exists$.

If $\int_{[-\varepsilon,\varepsilon]} |x|\, d\nu(x) < \infty$

$\Rightarrow$ the form is. $i\theta c - \frac{\sigma^2}{2}\theta^2 + \int (e^{i\theta x} - 1)\, d\nu(x)$

$\nu(-\varepsilon,\varepsilon) < \infty \underset{\not\Leftarrow}{\Rightarrow} \int_{[-\varepsilon,\varepsilon]} |x|\, d\nu(x) < \infty \underset{\not\Leftarrow}{\Rightarrow} \int_{[-\varepsilon,\varepsilon]} x^2\, d\nu(x) < \infty.$

If, in addition, $\nu[-\varepsilon,\varepsilon] < \infty$,

$\Rightarrow$ the form is.

$$\underbrace{i\theta c - \frac{\sigma^2}{2}\theta^2 + \nu(\mathbb{R}) \int (e^{i\theta x} - 1)\, \underbrace{\overbrace{\frac{d\nu(x)}{d\nu(\mathbb{R})}}^{dF}}_{\text{Probability}}}_{\lambda \mathbb{E}(e^{ix}-1)}$$

* $N \sim \text{Poisson}(\lambda)$, $X_k \sim F$ ind.

$\mathbb{E}\{e^{i\theta \sum_{k=1}^{N} X_k}\} = e^{\lambda \mathbb{E}(e^{ix}-1)}$.

* X is infinitely divisible

$\Leftrightarrow \exists\, \mu, \sigma^2, \nu$ (σ^2, ν ! ; $\mu = \mu(\varepsilon)$)

$\cdot\, \ni\cdot\, \mathbb{E}(e^{i\theta x}) = e^{i\theta\mu - \frac{\sigma^2}{2}\theta^2 + \int_{\mathbb{R}} (e^{i\theta x} - 1 - i\theta x\, \mathbb{I}(|x| \le \varepsilon))\, d\nu(x)}$

- If $\int (|x| \wedge 1)\, d\nu(x) < \infty$,

$\Rightarrow \mathbb{E}(e^{i\theta x}) = e^{i\theta c - \frac{\sigma^2}{2}\theta^2 + \int_{\mathbb{R}} (e^{i\theta x} - 1)\, d\nu(x)}$.

- If $\nu(\mathbb{R}) < \infty$, $\Rightarrow \exists\, \lambda > 0$, density function F

$\Rightarrow \mathbb{E}(e^{i\theta x}) = e^{i\theta c - \frac{\sigma^2}{2}\theta^2 + \lambda \int_{\mathbb{R}} (e^{i\theta x} - 1)\, dF(x)}$

- In the last case, $X \sim Y + Z$, $Y \sim N(c, \sigma^2)$
- If $\nu(-\infty, 0) = 0$, $Z = \sum_{k=1}^{N_\lambda} Z_i$, $N_\lambda \sim \text{Poisson}(\lambda)$

$\Rightarrow \exists$ a Laplace-Stieltjes $Z_i \sim F$ ind. transform, and this is true if we replace $i\theta$ by $-\theta$ throughout. $\theta \geq 0$.

- If $\nu(0,\infty) = 0$, $\Rightarrow$ similarly replace $i\theta$ by θ for $\theta \geq 0$.

Viernes 5 feb 2010.

Compound Poisson

$N_t \to$ Poisson process with rate λ.

$\{X_i\}_{i\in\mathbb{N}}$ iid $\perp \{N_t \mid t\geq 0\}$.

$$X(t) = \sum_{i=1}^{N(t)} X_i$$

$N \to$ Poisson random measure on

$$(\mathbb{R}^+\cup\{0\}\times\mathbb{R},\ \mathcal{B}(\mathbb{R}^+\cup\{0\}\times\mathbb{R}),\ \lambda(\ell\otimes F))$$

$$X(t) = \int_{(0,t]}\int_{\mathbb{R}} x\, dN(s,x)$$

$\to$ $\{X(t) \mid t\geq 0\}$ has stationary independent increments.

$$* \quad X(t) = \int_{(0,t]}\int_{\mathbb{R}} x\, dN(s,x) = \int f\, dN$$

$$= \underbrace{\int f^+ dN}_{\substack{<\infty \text{ a.s.} \\ \Leftrightarrow \int(1-e^{-f^+})d\mu<\infty}} - \underbrace{\int(-f^-)dN}_{\substack{<\infty \text{ a.s.} \\ \Leftrightarrow \int(1-e^{-f^-})d\mu<\infty}}$$

independent (just reminder).

- When either $\int(1-e^{-f^+})d\mu<\infty$ or $\int(1-e^{f^-})d\mu<\infty$,

$$\Rightarrow \mathbb{E}\{e^{i\theta\int f dN}\} = e^{\int(e^{i\theta f}-1)\,d\mu}.$$

$$* \quad \mathbb{E}\{e^{i\theta X(t)} \mid N(t)=n\} = \mathbb{E}\{e^{i\theta\sum_{k=1}^{n} X_k}\}$$

$$= [\mathbb{E}\{e^{i\theta x}\}]^n$$

$$\Rightarrow \mathbb{E}(e^{i\theta X(t)}) = \mathbb{E}\{[\mathbb{E}(e^{i\theta x})]^{N(t)}\}$$

$$\mathbb{E}\{z^{N(t)}\} = e^{-\lambda t(1-z)} = e^{\lambda t(z-1)} \quad \forall z\in\mathbb{C}$$

$$= e^{\lambda t(\mathbb{E}(e^{i\theta x})-1)}$$

$$\therefore \quad \mathbb{E}\{e^{i\theta X(t)}\} = \exp\{\lambda t\int_{\mathbb{R}}(e^{i\theta x}-1)dF(x)\}$$

$$* \quad \mathbb{E}\{e^{i\theta\int\int_{(0,t]\times\mathbb{R}} x\, dN(s,x)}\} = \exp\Big[\int\int_{\mathbb{R}^+\cup\{0\}\times\mathbb{R}} \underbrace{(e^{ix\,\mathbb{I}_{(0,t]}(s)}-1)}_{(e^{ix}-1)\,\mathbb{I}_{(0,t]}(s)}\, d(\lambda\ell\otimes F)\Big]$$

$$= \exp\Big[\int_{(0,t]}\int_{\mathbb{R}}(e^{ix}-1)\lambda\, dF(x)\,ds\Big] = e^{\lambda t\int(e^{i\theta x}-1)dF(x)}$$

• Assume $\mathbb{E}(|X|) = \int |x|\, dF(x) < \infty$.

$$\mathbb{E}(X(t)) = \mathbb{E}\left\{\sum_{k=1}^{N(t)} X_k\right\} = \mathbb{E}\{N(t)\}\,\mathbb{E}\{X_i\}$$

$$= \lambda t\, \mathbb{E}(X) = \lambda t \int x\, dF(x)$$

$$\mathbb{E}\left\{\int_{[0,t]}\int_{\mathbb{R}} x\, dN(s,x)\right\} = \int_{[0,t]}\int_{\mathbb{R}} x\, d\,\lambda\ell \otimes F(s,x).$$

$$= \lambda \int_{[0,t]}\int_{\mathbb{R}} x\, dF(x)\, ds = \lambda t \int_{\mathbb{R}} x\, dF(x)$$

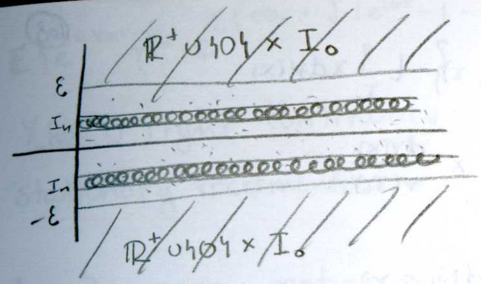
$$\mathbb{E}\left\{e^{i\theta(X(t) - \mathbb{E}(X(t)))}\right\} = e^{-i\theta\lambda t \int_{\mathbb{R}} x dF(x)}\, e^{\lambda t \int (e^{i\theta x} - 1) dF(x)}.$$

$$= e^{i\theta t \lambda \int (e^{i\theta x} - 1 - i\theta x)\, dF(x)}.$$

$$= e^{i\theta t \int_{\mathbb{R}\setminus\{0\}} (e^{i\theta x} - 1 - i\theta x)\, d(\lambda F(x))}$$

Let ν be the Lévy measure; $\nu(\{0\}) = 0$, $\int_{\mathbb{R}} (x^2 \wedge 1)\, d\nu(x)$

Let N be a Poisson random measure on $(\mathbb{R}^+ \cup \{0\} \times \mathbb{R}, \mathcal{B}(\mathbb{R}^+ \cup \{0\} \times \mathbb{R}), \ell \otimes \nu)$



; $I_0 = [-\varepsilon, \varepsilon]^c$

We make $I_n = [-\frac{\varepsilon}{n}, -\frac{\varepsilon}{n+1}) \cup (\frac{\varepsilon}{n+1}, \frac{\varepsilon}{n}]$

$\nu(I_n) < \infty \;\forall n.$

$N_n(A) = N(A \cap (\mathbb{R}^+ \cup \{0\} \times I_n))$

disjoint

They are independent Poisson random measures.

* $$\mathbb{E}\left\{e^{i\theta \int_{[0,t]}\int_{\mathbb{R}} x\, dN_n(s,x)}\right\} = e^{t\int_{I_n} (e^{i\theta x} - 1)\, d\nu(x)}$$

$$= e^{\nu(I_n) t \int (e^{i\theta x} - 1) \frac{d\nu(x)}{\nu I_n}}$$

if $\nu(I_n) > 0$

$\int_{[0,t]}\int_{\mathbb{R}} x\, dN_n(s,x)$ is a compound Poisson with jump rate $\lambda = \nu(I_n)$ and jump distribution

$$F(x) = \frac{\nu((-\infty, x] \cap I_n)}{\nu(I_n)}$$

- $\mathbb{E}\left\{\int_{[0,t)}\int_{\mathbb{R}} x\, dN_n(s,x)\right\} - t\int_{I_n} x\, d\nu(x)$

$$= t\nu(I_n)\int_{I_n} x\,\frac{d\nu(x)}{\nu(I_n)}$$

* $X_n(t) = \underbrace{\int_{[0,t)}\int_{I_n} x\, dN(s,x)}_{\int_{\mathbb{R}} x\, dN_n(s,x)}$

$$\mathbb{E}\left\{e^{i\theta\{X_n(t) - \mathbb{E}(X_n(t))\}}\right\}_{n\geq 1} = e^{t\int_{I_n}(e^{i\theta x} - 1 - i\theta x)\,d\nu(x)}$$

$$\int |x|\, d\nu(x) \leq \frac{1}{n}\nu(I_n) < \infty$$

$$\mathbb{E}\left\{e^{i\theta X_0(t)}\right\} = e^{t\int_{I_0}(e^{i\theta x} - 1)\,d\nu(x)}.$$

* $Y(t) = ct + X_0(t) + \sum_{k\in\mathbb{N}}\left(X_k(t) - \mathbb{E}(X_k(t))\right)$

$$= ct + \int_{[0,t]}\int_{[-\varepsilon,\varepsilon]^c} x\, dN(s,x) + \int_{[0,t]}\int_{[-\varepsilon,\varepsilon]} x\,(N(s,x) - \ell\otimes\nu(s,x))$$

* $\mathbb{E}\left\{e^{i\theta Y(t)}\right\} = e^{t\left(i\theta c + \int\left(e^{i\theta x} - 1 - i\theta x\,\mathbb{I}_{[-\varepsilon,\varepsilon]}(x)\right)d\nu(x)\right)}$

* $Y(t)$ is right continuous. process with stationary increments.

Def → Brownian motion ▷ Z is a Brownian motion if it has stationary increments,

$Z(t+s) - Z(s) \sim$ Normal $(?,?)$

$$Z(t) \xrightarrow[t\downarrow 0.]{P} 0$$

↳ Necessarily, $\exists\ \mu$ (drift) and σ^2 (diffusion coeficient) $\ni$.

$Z(t+s) - Z(s) \sim N(\mu t, \sigma^2 t)$.

* $Z(1) = Z(n\cdot\frac{1}{n}) = \sum_{k=1}^{n}\left(Z(\frac{k}{n}) - Z(\frac{k-1}{n})\right)$

$\mathbb{E}(Z(1)) = n\,\mathbb{E}(Z(\frac{1}{n}))$

$\mathrm{Var}(Z(1)) = n\,\mathrm{Var}(Z(\frac{1}{n}))$

$\mathbb{E}\{Z(\frac{k}{n})\} = k\,\mathbb{E}\{Z(\frac{1}{n})\} = \frac{k}{n}\underbrace{\mathbb{E}(Z(1))}_{\mu.}$

$\mathrm{Var}\{Z(\frac{k}{n})\} = k\,\mathrm{Var}\{Z(\frac{1}{n})\} = \frac{k}{n}\underbrace{\mathrm{Var}(Z(1))}_{\sigma^2}$

$\Rightarrow \forall\ q\in\mathbb{Q},\quad \mathbb{E}\{Z(q)\} = q\mu$

$\mathrm{Var}\{Z(q)\} = q\sigma^2.$

* There is a version of Brownian motion that is continuous.

$X(t)$ version $Y(t)$

$\mathbb{P}(X(t) = Y(t)) = 1 \ \forall t$

$\mathbb{P}(X(t_n) = Y(t_n) ; n \in \mathbb{N}) = 1$

$\exists$ process $Z(t)$ with continuity w.p. 1 and has stationary indep. increments (it has to have the right normal distribution).

* $\mathbb{E}\{e^{i\theta Z(t)}\} = e^{i\theta\mu t - \frac{\sigma^2 t}{2}\theta^2}$
$= e^{(i\theta\mu - \frac{\sigma^2}{2}\theta^2)t}$

* $Z(t) \leftarrow$ Brownian motion

$Y(t) \leftarrow$ The process that we built from the Poisson random measure.

$Z(\cdot) \perp Y(\cdot)$.

$\Rightarrow X(t) = Z(t) + Y(t)$ is the most general right continuous process with stationary independent increments.

$Z(t) + \int_{[0,t]}\int_{[-\varepsilon,\varepsilon]^c} x \, dN(s,x) + \int_{[0,t]}\int_{[-\varepsilon,\varepsilon]} x \, d(N - \ell\otimes\nu)(s,x)$

$= X(t)$

* $\mathbb{E}\{e^{i\theta X(t)}\} = e^{t\,\varphi(\theta)}$;

$\varphi(\theta) = ic\theta - \frac{\sigma^2}{2}\theta^2 + \int (e^{i\theta x} - 1 - i\theta x \,\mathbb{I}_{[-\varepsilon,\varepsilon]}(x))\, d\nu(x)$

Def $\to$ Lévy process

1) Stationary independent increments.

2) $X(t) \xrightarrow[t \downarrow 0]{P} 0$

* $\exists$ a version that is right continuous and has left limit.

$\lim_{s \downarrow t} X(s) = X(t)$; $\exists$?

* If $\int_{(-\varepsilon,\varepsilon)} |x|\, d\nu(x) < \infty$, $\sigma^2 = 0$

$$\varphi(\theta) = i\tilde{c}\theta + \int (e^{i\theta x} - 1)\, d\nu(x);$$

$$\tilde{c} = c - \int_{(-\varepsilon,\varepsilon)} x\, d\nu(x).$$

$$= i\tilde{c}\theta + \int_{(0,\infty)} (e^{i\theta x} - 1)\, d\nu(x) + \int_{(-\infty,0)} (e^{i\theta x} - 1)\, d\nu(x).$$

In this case,

$$X(t) = \tilde{c}t + \underbrace{\int_{[0,t)}\int_{(0,\infty)} x\, dN(s,x)}_{\geq 0 \text{ nondecreasing}} + \underbrace{\int_{[0,t)}\int_{(-\infty,0)} x\, dN(s,x)}_{\geq 0 \text{ non increasing}}$$

So, when $\sigma^2 = 0$, $\int_{[-\varepsilon,\varepsilon]} x\, d\nu(x) < \infty$,

$$X(t) = \tilde{c}(t) + X_1(t) - X_2(t),$$

where $X_1(t)$ independent non decreasing Lévy processes

(?? Pero dijimos que uno era no creciente y el otro no decreciente...).

* If $\nu(-\infty,0) = 0$, then we say that there are no negative jumps

↳ spectrally positive.

Analogously, if $\nu(0,\infty) = 0$, we say that there are no positive jumps (spectrally negative).

$$\mathbb{E}\{e^{-\theta X(t)}\} = e^{t\varphi(\theta)};$$

$$\varphi(\theta) = -c\theta + \frac{\sigma^2}{2}\theta^2 + \int_{(0,\infty)} (e^{-\theta x} - 1 + \theta x \mathbb{I}_{(0,\varepsilon]}(x))\, d\nu(x).$$

* $\sigma^2 = 0$, $\int_{[-\varepsilon,\varepsilon]} |x|\, d\nu(x) < \infty$ ← bounded variation on finite intervals w.p. 1.

otherwise, unbounded variation

* if $\sigma^2 = 0$, $\nu(-\infty,\infty) = 0$, and $\int_{(0,\infty)} x\, d\nu(x) < \infty$,

$$X(t) = \tilde{c}t + \int_{[0,t]}\int_{\mathbb{R}^+} x\, dN(s,x)$$

If, in addition, $\tilde{c} \geq 0$, $\Rightarrow X(t)$ is nondecreasing.

* Non decreasing Lévy processes are called subordinators

↳ $\mathbb{E}\{e^{i\theta X(t)}\} = e^{t\varphi(\theta)}$;

$$\varphi(\theta) = -(\tilde{c}\theta + \int (1 - e^{-\theta x})\, d\nu(x))$$

* Reminder: let N be a Poisson random measure.

If $\mu(A) = \infty$, $\Rightarrow \mathbb{P}(N(A) = \infty) = 1$.

If $t\nu(0, \varepsilon] = \infty$

$\longleftarrow = \ell \otimes \nu((s, s+t] \times (0, \varepsilon])$

$N((s, s+t] \times (0, \varepsilon]) = \infty$ wp1

Similarly, this happens for $[-\varepsilon, 0)$

* When $\tilde{c} = 0$, the subordinator is called a $\boxed{\text{pure jump}}$ process

* If $\nu([-\varepsilon, \varepsilon]) < \infty \Rightarrow 0 < \nu(\mathbb{R}) < \infty$

$\Rightarrow \int_{[-\varepsilon, \varepsilon]} |x|\, d\nu(x) < \infty$.

$\sigma^2 = 0$

$$\varphi(\theta) = \tilde{c}\theta + \int (e^{i\theta x} - 1)\, d\nu(x)$$

$$= \tilde{c}\theta + \underbrace{\nu(\mathbb{R})}_{\lambda} \int (e^{i\theta x} - 1)\, \underbrace{\frac{d\nu(x)}{\nu(\mathbb{R})}}_{F}$$

$\Rightarrow X(t)$ is a compound Poisson process $+ \tilde{c}t$.

If $\nu = 0 \Rightarrow$ we have a Brownian motion

If $\nu = 0, c = 0, \sigma^2 = 0, \Rightarrow X(t) = 0$.

* Let $X(t)$ be a Levy process with no negative jumps.

$$\varphi(\theta) = -c\theta + \frac{\sigma^2}{2}\theta^2 + \int (e^{-\theta x} - 1 + \theta x\, \mathbb{I}_{(0,\varepsilon]}(x))\, d\nu(x)$$

$\mathbb{E}\{e^{-\theta X(t)}\} = e^{\varphi(\theta)t}$.

* For a subordinator,

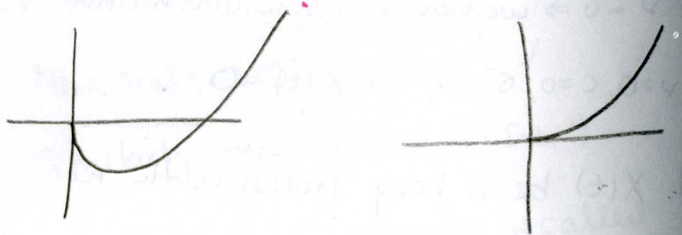
$$\varphi(\theta) = -\left(\tilde{c}\theta + \int_{\mathbb{R}^+} (1-e^{-\theta x})\,d\nu(x)\right) \quad ; \tilde{c} \geq 0$$

$$\int_{(0,\varepsilon]} x\,d\nu(x) < \infty.$$

* If its not a subordinator,

$$\Rightarrow \varphi(\theta) \xrightarrow[\theta\to\infty]{} \infty, \quad \varphi(0)=0,$$

φ is convex ($\Rightarrow$ continuous).



$$\varphi'(0) = -\left(c + \int_{(\varepsilon,\infty)} x\,d\nu(x)\right)$$

$$\varphi''(0) = \sigma^2 + \int_{(0,\infty)} x^2\,d\nu(x)$$

$$\varphi^{(n)}(0) = (-1)^n \int_{(0,\infty)} x^n\,d\nu(x)$$

* We define

$$\psi(0) = \lim_{\beta\downarrow 0} \psi(\beta) = \inf\{\theta \mid \varphi(\theta) > 0\}.$$

$\leftarrow$ inverse function of φ.

* $\mathbb{E}\{e^{-\alpha X(t)}\} = e^{-t\varphi(\alpha)}$

$$\mathcal{F}_s = \sigma(X(s) \mid 0 \leq s \leq t)$$

$$\mathcal{F}_s = \bigcap_{u>s} \mathcal{F}_u$$

$A \in \mathcal{F}$

$\mathbb{P}(A) = 0$

$B \subseteq A \Rightarrow B \in \mathcal{F}_0$

* $\mathbb{E}\{e^{-\theta X(t)} \mid \mathcal{F}_s\} = \mathbb{E}\{e^{-\theta(X(t)-X(s))}\, e^{-\theta X(s)} \mid \mathcal{F}_s\}$

$$= \mathbb{E}\{e^{-\theta(X(t)-X(s))} \mid \cancel{\mathcal{F}_s}\}\, e^{-\theta X(s)}$$

$$= e^{\varphi(\theta)(t-s)}\, e^{-\theta X(s)}$$

* $\mathbb{E}\{e^{-\theta X(t) - t\varphi(\theta)} \mid \mathcal{F}_s\} = e^{-\theta X(s) - t\varphi(\theta)}$

Theorem → optional stopping

→ $\{\mathcal{F}_t \mid t \geq 0\}$, $\mathcal{F}_s \subseteq \mathcal{F}_t \; \forall s \leq t$

$$\mathcal{F}_t = \bigcap_{s > t} \mathcal{F}_s$$

$\mathcal{F}_0$ contains all B s.t. $B \subseteq A \in \mathcal{F}$, $\mathbb{P}(A) = 0$.

$M(t)$ ← right continuous

s.t. $\mathbb{E}\{M(t) \mid \mathcal{F}_s\} = M(s)$,

$\mathbb{E}\{|M(t)|\} < \infty$.

Let τ be a nonnegative r.v.;

$\{\tau \leq t\} \in \mathcal{F}_t$

M is called a martingale,

τ is called a stopping time.

$\Rightarrow \forall t \in \mathbb{R}^+ \cup \{0\}$, $\mathbb{E}\{M(t \wedge \tau)\} = \mathbb{E}\{M(0)\}$.

* Make $\tau_a = \inf\{t \mid X(t) = -a\}$

$$e^{-\theta X(t) - \varphi(\theta) t}$$

$$\boxed{\varphi(\psi(\beta)) = \beta}$$

$e^{-\psi(\beta) X(t) - \beta t}$ ← martingale $\forall \beta > 0$

$$1 = \mathbb{E}\{e^{-\psi(\beta) X(\tau_a \wedge t) - \beta(\tau_a \wedge t)}\}$$

$$X(\tau_a \wedge t) \geq -a \; ; \quad \tau_a \wedge t \geq 0$$

$$e^{-\psi(\beta) X(\tau_a \wedge t) - \beta(\tau_a \wedge t)} \leq e^{\psi(\beta) a}$$

On $\tau_a = \infty$,

$$e^{-\psi(\beta) X(\tau_a \wedge t) - \beta(\tau_a \wedge t)} \leq e^{\psi(\beta) a} e^{-\beta t} \xrightarrow[t \to \infty]{} 0$$

$$\mathbb{E}\{e^{-\psi(\beta) X(\tau_a \wedge t) - \psi(\beta)(\tau_a \wedge t)}\}$$

$$\xrightarrow[t \to \infty]{} \mathbb{E}\left(e^{-\psi(\beta) \underbrace{X(\tau_a)}_{-a} - \psi(\beta)\tau_a} \mathbb{I}_{\{\tau_a < \infty\}}\right)$$

$$\mathbb{E}\{e^{-\beta \tau_a}\} = e^{-\psi(\beta) a} \xrightarrow[\beta \downarrow 0]{} \mathbb{P}(\tau_a < \infty) = \begin{cases} 1 & \varphi'(0) \geq 0 \\ e^{-\psi(0)} & \varphi'(0) < 0. \end{cases}$$

$$\psi(\beta) = \tau\beta + \int_{\mathbb{R}^+} (1 - e^{-\beta x})\, d\tilde{\nu}(x).$$