* $\mathbb{E}\left\{e^{-\sum_{i=1}^{k}\theta_i \int f_i \, d\tilde{N}_i}\right\}$

Note → N lives on $(G\times H;\ \mathcal{G}\otimes\mathcal{H})$

$\int f_i \, d\tilde{N}_i = \int f_i(x)\, \mathbb{I}_{B_i}(y)\, dN(x,y)$

$\int \mathbb{I}_A(x)\, \mathbb{I}_{B_i}(y)\, dN(x,y) = \int \mathbb{I}_{A\times B_i}\, dN$

$= N(A\times B_i) = \tilde{N}_i(A) = \int \mathbb{I}_A \, d\tilde{N}_i$

$$= \mathbb{E}\left\{e^{-\sum_{i=1}^{k}\underbrace{\theta_i \int f_i(x)\, \mathbb{I}_{B_i}(y)}_{\geq 0,\ \text{Borel}\ \forall x,y}\, dN(x,y)}\right\}$$

$(G;\mathcal{G}) \xrightarrow{f_i(x)} (\mathbb{R}, \mathcal{B}(\mathbb{R}))$

$(\ ,\mathcal{H}) \xrightarrow{\mathbb{I}_B(y)} (\mathbb{R}, \mathcal{B}(\mathbb{R}))$

$\Rightarrow (G\times H;\ \mathcal{G}\otimes\mathcal{H}) \xrightarrow{f_i(x)\,\mathbb{I}_B(y)} (\mathbb{R}, \mathcal{B}(\mathbb{R}))$

$$= e^{-\int \left(1 - e^{-\sum_{i=1}^{k}\theta_i f_i(x)\, \mathbb{I}_{B_i}(y)}\right) d\mu(x,y)}$$

$$= e^{-\int \sum_{i=1}^{k}\left(1 - e^{-\theta_i f_i(x)}\right)\mathbb{I}_{B_i}(y)\, d\mu(x,y)}$$

$$= e^{-\sum_{i=1}^{k}\int\left(1 - e^{-\theta_i f_i(x)}\right) d\tilde{\mu}_i(x)}$$

$$= \prod_{i=1}^{k} e^{-\int\left(1 - e^{-\theta_i f_i(x)}\right) d\tilde{\mu}_i}$$

★ As before, this implies that $\{\int f_i \, d\tilde{N}_i\}_{i=1}^{k}$ are ind r.v., & that $\tilde{N}_i$ are Poisson random measures with average measures $\tilde{\mu}_1, \dots, \tilde{\mu}_k$.

Theorem → Let N be a Poisson random measure on $(G\times H,\ \mathcal{G}\otimes\mathcal{H})$. Assume that $\tilde{\mu}(A) = \mu(A\times H)$ is σ-finite, and that $B_1, \dots, B_k \in \mathcal{H}$ are disjoint. We define $\tilde{N}_1, \dots, \tilde{N}_k$ as $\tilde{N}_i(A) = N(A\times B_i)$,

$\Rightarrow \tilde{N}_1, \dots, \tilde{N}_k$ are independent Poisson random measures with average measures $\mu_1, \dots, \mu_k$ where $\tilde{\mu}_i(A) = \mu(A\times B_i)$.

$\mathbb{P}\{(\Omega_{ij}, X_{ij}) \in A\} = \int\int \mathbb{I}_A(x,y)\, \frac{\mu(dx)}{\mu(A_i)}\, \mathbb{P}(x,dy)$

Let N be a Poisson random measure with average measure μ on $(G,\mathcal{G})$ and, for each point x, we pick a point in H according to some distribution $\mathbb{P}(x,A)$. Then, we get a Poisson random measure on $(G\times H,\ \mathcal{G}\otimes\mathcal{H})$ with average measure λ, where $\lambda(A\times B) = \int_{x\in A}\int_{y\in B} \mathbb{P}(x,dy)\,\mu(dx)$

$= \int_A \mathbb{P}(x,B)\,\mu(dx)$

$$\lambda(A \times H) = \int_A \underbrace{\mathbb{P}(x, H)}_{1} \mu(dx) = \mu(A).$$

* For example, if N_t is a Poisson process with rate $0 < \lambda < \infty$ of arrivals, we independently decide to send it in direction i with probability p_i $\left(\sum_{i=1}^{k} p_i = 1\right)$, this is the same as having a Poisson random measure on $([0,\infty), \mathcal{B}([0,\infty)))$ with average measure $\lambda \ell$

$$G = [0, \infty), \quad \mathcal{G} = \mathcal{B}([0,\infty))$$

$$H = \{1, \dots, k\} \quad \mathcal{H} = 2^H$$

$$\mathbb{P}(x, \{i\}) = p_i \ \forall x.$$

$$\lambda(A \times \{i\}) = \int_A \underbrace{\mathbb{P}(x, \{i\})}_{p_i} d\mu(x)$$

$$= p_i \mu(A) = p_i \lambda \ell(A) = \lambda p_i \ell(A).$$

Conclusion → $\tilde{N}_1, \dots, \tilde{N}_k$ are ind. Poisson random measures with average measures $p_i \ell(A)$

$\therefore \tilde{N}_{i,t} = \tilde{N}_i([0,t))$ are independent Poisson processes with rates $\lambda p_1, \dots, \lambda p_k$

* If N_t is a Poisson process with rate λ and $p(x)$ is an arbitrary nonnegative Borel function (with values in $[0,1]$)

$$\xrightarrow{p(x)} 1 \qquad \mathbb{P}(x, \{1\}) = p(x)$$

$$\xrightarrow{1-p(x)} 2 \qquad \mathbb{P}(x, \{2\}) = 1 - p(x).$$

$$\int_A \mathbb{P}(x, B) \mu(dx) = \int_A \mathbb{P}(x, \{1\}) \lambda \ell(dx)$$

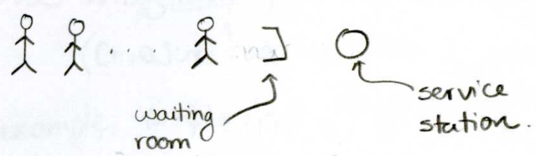
$$= \lambda \int_A \mathbb{P}(x) dx.$$

$\tilde{N}_{i,t} \triangleq$ # of arrivals that were chosen in direction i, $i \in \{1, 2\}$.

$\tilde{N}_{it} \triangleq$ Non-homogeneous Poisson process with rate function $\lambda(x) = \lambda \mathbb{P}(x)$

If $\lambda(t) \geq 0$ is a Borel function, bounded by λ,
$\Rightarrow$ taking $\mathbb{P}(x) = \frac{\lambda(x)}{\lambda}$, we can use splitting to construct a non-homogeneous Poisson process with rate function $\lambda(x)$ from a homogeneous Poisson process with rate λ.

Queing theory



Arrival / service / service / # of servers.

Assume that the arrivals together with the service time (jointly) form a Poisson random measure.

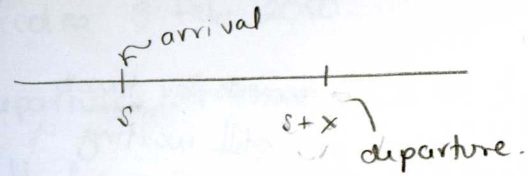
Alternatively, the arrival time forms a Poisson random measure, and the service time of an arrival at time t is chosen according to some distribution $\mathbb{P}(t, B)$. $F_t(x)$

Let λ be the average arrival measure. The overall measure is defined as $\mu(A \times [0,x])$

$= \int_A F_t(x)\, \lambda(dt)$.

$G = [0,\infty)$, $\mathcal{G} = \mathcal{B}([0,\infty))$

$H = [0,\infty)$, $\mathcal{H} = \mathcal{B}([0,\infty))$.



At a given time t, we have 3 options for any given customer:

$A_1 = \{(s,x) \mid s > t,\ x \geq 0\}$

The customer arrived (and left) after t.

$A_2 = \{(s,x) \mid s \leq t,\ s+x > t\}$

The customer arrived before (or at) t and left after t.

$A_3 = \{(s,x) \mid s+x \leq t\}$

The customer had already been serviced (and left) at t.

A_1, A_2, A_3 disjoint.

$N_i(A) = N(A \cap A_i)$ ← indep. rand. measures with av. measures $\mu_i(A)$ $= \mu(A \cap A_i)$

In particular,

$N_2([0,t]\times\mathbb{R}) \leftarrow$ # of arrivals by time t that are still waiting

$N(A_2) = N_2([0,t]\times\mathbb{R})$.

Similarly,

$N_3([0,t]\times\mathbb{R}) \leftarrow$ # of customers that had already left by time t.

↳ This implies that the number of departures and the number of customers at the system at time t. are ind. r.v. and distributed Poisson with parameters $\mu(A_2)$, $\mu(A_3)$ respectively.

(xt!)

* $\mu(A_2) = \int_{[0}^{t]} (1 - F_s(t-s))\, \lambda(ds)$.

$\mu(A_3) = \int_{[0}^{t]} F_s(t-s)\, \lambda(ds)$.

Miércoles 3·Feb·2010.

departures at time t $\perp$ # of customers present at time t.

Both have a Poisson distribution.

* Let N be a Poisson random measure on $(G, \mathcal{G}, \mu)$; μ σ-finite

↳ always from now on

$g : (G, \quad) \rightarrow (H, \mathcal{H})$ measurable

$\mathcal{N}(A) = N(g^{-1}(A))$

If $A_1, \ldots, A_k$ disjoint, $\Rightarrow \{g^{-1}(A_i)\}_{i=1}^{k}$ disjoint

$\Rightarrow \{N(g^{-1}(A_i))\}_{i=1}^{k}$ ind ; $N(g^{-1}(A)) \sim \text{Poisson}(\mu(g^{-1}(A)))$

* $d(s,x) = s+x$

$N_d \leftarrow$ random measure on $([0,\infty), \mathcal{B}([0,\infty)))$

↳ actually, a Poisson random measure.

$\mu_d(A) = ?$ $\quad F_t(x)$ $\quad \Lambda(t) = \mu([0,t]\times[0,\infty))$

$\Lambda_d(t) = \mu_d([0,t]) = \int_{[0,t]} F_s(t-s)\, \Lambda(ds)$

$\mu_d(A) = \int_A \Lambda_d(dt)$ $\qquad$ * $\Lambda_d(t) \leq \Lambda(t)$

$\Rightarrow \mu_d$ is σ-finite

$N_{d,t} = N_d([0,t]) \Rightarrow$ generalized non-homogeneous Poisson process.

* Departure process up to time t is a non-homogeneous Poisson process and is independent of the number of customers in the system at time t.

* If $\frac{\Lambda(t)}{t} \xrightarrow[t\to\infty]{} \lambda$ and $F_s = F$

$$\Lambda_d = \int_{[0,t]} F(t-s)\, d\Lambda(s) = \iint_{0\le s+x\le t} \underbrace{dF(x)\, d\Lambda(s)}_{d\mu(s,x)}$$

$$= \iint_{0\le s+x\le t} d\Lambda(s)\, dF(x) = \int_{[0,t]} \Lambda(t-x)\, dF(x)$$

$$= \mathbb{E}\{\Lambda(t-x)\, \mathbb{I}_{\{x\le t\}}\} \; ; \; X \sim F$$

$$\Rightarrow \mathbb{E}\{\Lambda(t-x)\, \mathbb{I}_{\{x\le \varepsilon\}}\} \le \underbrace{\mathbb{E}\{\Lambda_{(t-x)}\, \mathbb{I}_{\{x\le t\}}\}}_{=\Lambda_d(t) \le \Lambda(t)}$$

$$\ge \mathbb{E}\{\Lambda(t-\varepsilon)\, \mathbb{I}_{\{x\le\varepsilon\}}\} = \Lambda(t-\varepsilon) F(\varepsilon)$$

If, in addition, $\frac{\Lambda(t)}{t} \to \lambda$

$$\Rightarrow \underbrace{\frac{\Lambda(t-\varepsilon)}{t}}\, F(\varepsilon) \le \frac{\Lambda_d(t)}{t} \le \frac{\Lambda(t)}{t}$$

$$= \frac{\Lambda(t-\varepsilon)}{t-\varepsilon} \underbrace{\left(1 - \frac{\varepsilon}{t}\right)}_{= \frac{t-\varepsilon}{t}} F(\varepsilon)$$

- If $\lambda = \infty$, take $\varepsilon > 0$ for which $F(\varepsilon) > 0$ and, then, necessarily $\frac{\Lambda_d(t)}{t} \to \infty$.

- If $\lambda = 0$, $\Rightarrow \frac{\Lambda_d(t)}{t} \xrightarrow[t\to\infty]{} 0$

- If $\lambda \in \mathbb{R}^+$, $\Rightarrow \lambda F(\varepsilon) \le \underline{\lim} \frac{\Lambda_d(t)}{t} \le \overline{\lim} \frac{\Lambda_d(t)}{t} \le \lambda$

$\forall \varepsilon > 0$.

$\Rightarrow$, as $\varepsilon \to \infty$, $\lambda \le \lim_{t\to\infty} \frac{\Lambda_d(t)}{t}$

Corollary $\Rightarrow$ If $F_s = F$, $\frac{\Lambda(t)}{t} \xrightarrow[t\to\infty]{} \lambda$

$$\Rightarrow \frac{\Lambda_d(t)}{t} \xrightarrow[t\to\infty]{} \lambda \quad \forall \lambda \in [0, \infty]$$

\# of customers at time t $\sim$ Poisson $(\Lambda(t) - \Lambda_d(t))$

$$\Lambda(t) - \Lambda_d(t) = \Lambda(t) - \int_{[0,t]} F_S(t-s)\, d\Lambda(s)$$

$$= \int_0^t (1-F(t-s))\lambda(s)\, ds = \int_0^t (1-F(s))\lambda(t-s)$$

Λ has a density $\lambda(s)$

No hay que tomar $\lambda(t+s)$ y cambiar el signo de la int?

When Λ has a density and $F_S = F$, $\Rightarrow$ the # of customers at time t

$$\sim \text{Poisson}\left(\int_0^t (1-F(s))\lambda(t-s)\, ds\right)$$

If $\lambda(s) = \lambda \Rightarrow \int_0^t (1-F(s))\lambda(t-s)\, ds$

$$= \lambda \int_0^t (1-F(s))\, ds.$$

Regardless, if $\lambda(s)$ is bounded, $\mathbb{E}(X) < \infty$, $X \sim F$

and $\lambda(t) \xrightarrow[t\to\infty]{} \lambda$,

$$\Rightarrow \int_0^t (1-F(s))\lambda(t-s)\, ds \to \lambda \int_0^\infty (1-F(s))\, ds.$$

$$\int_0^t (1-F(s))\lambda(t-s)\, ds = \int_0^\infty \underbrace{(1-F(s))\lambda(t-s)\, \mathbb{I}_{[0,s]}(t)}_{g(s,t) \xrightarrow[t\to\infty]{} \lambda(1-F(s))}\, ds.$$

$$= \lambda\, \mathbb{E}(X)$$

This follows from the dominated convergence theorem for σ-finite measures.

* $L(t) \rightarrow$ # of customers at t.

$$\mu(A \times B) = \int_A \lambda(s)\, ds \cdot \int_B dF(x).$$

- Arrivals form a non-homogeneous Poisson process with rate function $\lambda(s)$, Borel, nonnegative, bounded with $\lambda(t) \xrightarrow[t\to\infty]{} \lambda$; $\lambda \in \mathbb{R}^+$.

Service times are independent of the arrival process and have a finite mean $\mathbb{E}(X)$.

Then, $L(t) \xrightarrow{d} L$, where $L \sim \text{Poisson}(\lambda \mathbb{E}(X))$;

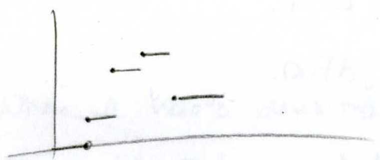
Notation $\rightarrow \rho = \lambda \mathbb{E}(X)$: traffic intensity

Compound Poisson process

$N_t \to$ Poisson process with rate λ; $\lambda \in \mathbb{R}^+$

$\{X_i\}_{i\in\mathbb{N}}$ iid (finite valued) r.v. indep. of $\{N_t \mid t \geq 0\}$.

Def $\to$ $\sum_{i=1}^{N_t} X_i$ is called a compound Poisson process



$S_1, S_2, \ldots \leftarrow$ arrival points of N_t.

$\hookrightarrow S_i = \inf\{t \mid N_t = i\}$

$S_1, \ldots, S_n, X_1, \ldots, X_n \mid N_t = n \sim tU_{\pi(1)}, \ldots, tU_{\pi(n)}, X_1, \ldots, X_n$.

$\pi \to$ random permutation that reorders $U_1, \ldots, U_n$.

$\hookrightarrow U_i \sim \mathrm{Unif}(0,1)$

$tU_i \sim \mathrm{Unif}(0,t)$.

$\mathbb{P}\{(tU_{\pi(1)}, \ldots, tU_{\pi(n)}, X_1, \ldots, X_n) \in A \mid U_1, \ldots, U_n\}$

For any fixed (non-random) permutation,
$(X_1, \ldots, X_n) \sim (X_{\pi(1)}, \ldots, X_{\pi(n)})$

The permutation π depends on the values of $U_1, \ldots, U_n$

$= \mathbb{P}\{(tU_{\pi(1)}, \ldots, tU_{\pi(n)}, X_{\pi(1)}, \ldots, X_{\pi(n)}) \in A \mid U\}$

We take $\mathbb{E}$ on both sides

$\Rightarrow tU_{\pi(1)}, \ldots, tU_{\pi(n)}, X_1, \ldots, X_n \sim tU_{\pi(1)}, \ldots, tU_{\pi(n)}, X_{\pi(1)}, \ldots, X_{\pi(n)}$.

$\Rightarrow (S_1, \ldots, S_n, X_1, \ldots, X_n) \sim (tU_{\pi(1)}, \ldots, tU_{\pi(n)}, X_{\pi(1)}, \ldots, X_{\pi(n)})$.

Let $Y \sim \mathrm{Poisson}(\lambda t)$

(tU_i, X_i) ; $U_i \sim \mathrm{Unif}(0,1)$, $X_i \sim F$.

iid random pairs with $U_i \perp X_i$.

$\sum_{i=1}^{N_t} X_i \sim \sum_{i=1}^{Y} X_i$.

We define $\mathcal{N}$, a Poisson random measure on

$(\mathbb{R}^+ \cup \{0\} \times \mathbb{R},\ \mathcal{B}(\mathbb{R}^+ \cup \{0\}) \otimes \mathcal{B}(\mathbb{R}))$

Jueves 4. feb. 2010.

Let N be a Poisson random measure on $(G, \mathcal{G}, \mu)$.

Let $f \geq 0$ be a Borel function.

$$\mathbb{E}\{e^{-\int f dN}\} = e^{-\int (1-e^{-f}) d\mu}$$

- $\int f dN < \infty$ a.s. $\iff \int (1-e^{-f}) d\mu < \infty$.

$\iff$ some (and then all) $\varepsilon > 0$,

$$\mu(f^{-1}((\varepsilon, \infty))) < \infty, \quad \int_{f^{-1}((0,\varepsilon])} f \, d\mu < \infty.$$

$$\mu_f((\varepsilon, \infty)) < \infty, \quad \int_{(0,\varepsilon]} x \, d\mu_f(x) < \infty.$$

- $\int (1-e^{-f}) d\mu = \int (1-e^{-x}) d\mu_f(x)$
- $\mathbb{E}(e^{-\theta \int f dN}) = e^{-\int (1-e^{-\theta f}) d\mu} = e^{-\int (1-e^{-\theta x}) d\mu_f(x)}$

- $$\int_{f^{-1}([a,b])} f \, dN = \int_{(a,b]} x \, dN_f(x)$$

* Under the above conditions, this is also an almost sure finite random variable.

- $\mathbb{E}(e^{-\theta \int_{f^{-1}(a,b]} f dN})$ $\qquad \mathbb{I}_{f^{-1}([a,b])}(x) = \mathbb{I}_{[a,b]}(f(x))$

$$= \mathbb{E}\{e^{-\theta \int f \, \mathbb{I}_{[a,b]}(f) \, dN}\}$$

$$= e^{-\int \underbrace{(1-e^{-\theta f \, \mathbb{I}_{[a,b]}(f)})}_{(1-e^{-\theta f}) \, \mathbb{I}_{[a,b]}(f).} d\mu}$$

$$= e^{-\int_{f^{-1}(a,b]} (1-e^{-\theta f}) d\mu}.$$

$$\Rightarrow \mathbb{E}\{e^{-\theta \int_{f^{-1}(a,b]} f dN}\} = e^{-\int_{f^{-1}(a,b]} (1-e^{-\theta f}) d\mu}.$$

$$= \exp\left\{-\int_{[a,b]} (1-e^{-\theta x}) d\mu_f(x)\right\}.$$

Let $f = \mathbb{I}_A$

$$\mathbb{E}(\underbrace{\int \mathbb{I}_A dN}_{N(A)}) = \mathbb{E}(N(A)) = \mu(A) = \int \mathbb{I}_A d\mu.$$

Let $f = \sum_{i=1}^{n} a_i \mathbb{I}_{A_i}$

$$\mathbb{E}\{\int f\, dN\} = \mathbb{E}\{\int \sum_{i=1}^{n} a_i \mathbb{I}_{A_i}\, dN\} = \sum_{i=1}^{n} a_i \mathbb{E}\{\int \mathbb{I}_{A_i}\, dN\}$$

$$= \sum_{i=1}^{n} a_i \int \mathbb{I}_{A_i}\, d\mu = \int \sum_{i=1}^{n} a_i \mathbb{I}_{A_i}\, d\mu. = \int f\, d\mu$$

Let $f_n \nearrow f$

$\Rightarrow \forall f \geq 0$ Borel function,

$$\underbrace{\mathbb{E}\{\int f\, dN\}}_{= \mathbb{E}\{\int_{[0,\infty)} x\, dN_f(x)\}} = \underbrace{\int f\, d\mu}_{= \int_{[0,\infty)} x\, d\mu_f(x).}$$

$f = f^+ + f^-$ $\quad \int f^+ dN = \int_{f^{-1}(0,\infty)} f\, dN$

$\int f^- dN = \int_{f^{-1}(-\infty,0)} f\, dN$

$$f^{-1}(0,\infty) \cap f^{-1}(-\infty,0) = f^{-1}\underbrace{((0,\infty)\cap(-\infty,0))}_{\phi} = \phi.$$

* Let $A_1, \ldots, A_k$ be disjoint

$\Rightarrow \{\int_{A_i} f\, dN\}$ are independent.

$$\Rightarrow \int_{f^{-1}(0,\infty)} f\, dN \perp \int_{f^{-1}(-\infty,0)} f\, dN.$$

If one of them is a.s. finite,

$\Rightarrow \int f dN$ is well defined. That is, $\int f dN$ is a well defined r.v. $\Leftrightarrow$ either

$$\int_{f^{-1}(0,\infty)} (1-e^{-f})\, d\mu < \infty \quad \text{or} \quad \int_{f^{-1}(-\infty,0)} (1-e^{-f})\, d\mu < \infty.$$

* Let $f \geq 0$ be a Borel function.

$$\mathbb{E}\{\int_A f\, dN\} = \int_A f\, d\mu < \infty.$$

$\int_A f\, dN$ a.s. finite

$$\mathbb{E}\{e^{-\theta(\int_A f dN - \int_A f d\mu)}\} = e^{\theta \int_A f d\mu}\, \mathbb{E}\{e^{-\theta \int_A f dN}\}$$

$$= e^{\theta \int_A f d\mu}\, e^{-\int_A (1-e^{-\theta f}) d\mu}$$

$$= \exp\{\int_A (e^{-\theta f} - 1 + \theta f)\, d\mu\}.$$

$$\Rightarrow \mathbb{E}\left[\exp\left\{-\theta\left(\int_{f^{-1}(B)} f\,dN - \mathbb{E}\left\{\int_{f^{-1}(B)} f\,dN\right\}\right)\right\}\right]$$

$$= \exp\left\{\int_{f^{-1}(B)} (e^{-\theta f} - 1 + \theta f)\,d\mu\right\}$$

$$= \exp\left\{\int_B (e^{-\theta x} - 1 + \theta x)\,d\mu_f(x)\right\}$$

$\mu_f((\varepsilon,\infty)) < \infty.$

$\int_{(0,\varepsilon]} x\,d\mu_f < \infty.$

* What if $\int_{(0,\varepsilon]} x\,d\mu_f = \infty$ for all $\varepsilon > 0$

but $\mu_f(\varepsilon,\infty) < \infty$.

However, let's assume that $\underbrace{\int_{(0,\varepsilon)} x^2\,d\mu_f}_{= \int_{f^{-1}(0,\varepsilon)} f^2 d\mu} < \infty$.

Let's take $0 < a < b < \infty$.

$\Rightarrow a^2 \mu_f(a,b] \le \int_{f^{-1}(a,b)} f^2 d\mu = \int_{(a,b)} x^2 d\mu_f(x) \le b^2 \mu_f((a,b])$

$\Rightarrow$, since, for some $\varepsilon > 0$,

$\int_{(0,\varepsilon)} x^2\,d\mu_f < \infty$ and $\mu_f((\varepsilon,\infty)) < \infty$, then

$\forall\ 0 < a < b < \infty$, $\int_{(a,b]} x^2 d\mu_f(x) < \infty$, $\mu_f((a,b)) < \infty$.

also, $a\mu_f((a,b]) \le \int_{(a,b]} x\,d\mu_f(x) \le b\mu_f(a,b]$,

so $\int_{(a,b]} x\,d\mu_f(x) < \infty$.

* Suppose $0 < a_n \downarrow 0$; $a_0 = \varepsilon$.

$$\Rightarrow \mathbb{E}\left\{e^{-\theta\left(\int_{(a_n,a_{n-1}]} x\,dN_f - \int_{(a_n,a_{n-1}]} x\,d\mu_f\right)}\right\}$$

$\int_{(a_n,a_{n-1}]} x\,d(N_f - \mu_f)$

indep. r.v. since $(a_n, a_{n-1}]$ are disjoint.

$$= \exp\left\{\int_{(a_n,a_{n-1}]} (e^{-\theta x} - 1 + \theta x)\,d\mu_f(x)\right\}.$$

• $\sum_{n\in\mathbb{N}} \int_{(a_n,a_{n-1}]} x\,d(N_f - \mu_f)$ ← Does this converge to a well defined r.v. ?

$$\mathbb{E}\left\{e^{-\theta\sum_{n=1}^{m} \int_{(a_n,a_{n-1}]} x\,d(N_f-\mu_f)}\right\} = \mathbb{E}\left\{e^{-\theta\int_{(a_m,\varepsilon]} x\,d(N_f-\mu_f)}\right\}$$

$$= \exp\left\{\overbrace{\int_{(a_m,\varepsilon]} (e^{-\theta x} - 1 + \theta x)\,d\mu_f(x)}^{\ge 0}\right\} \xrightarrow[m\to\infty]{\nearrow} e^{\int_{(0,\varepsilon]} (e^{-\theta x} - 1 + \theta x) d\mu_f(x)}.$$